Chapter 1

Gain vs Directivity

Some assumptions

1. The entire system is assumed to be linear (including the power amplifier and phase shifter)

2. Conductor and dielectric losses are assumed to be zero (especially, when compared to the $P_{\text{rad}}$)

From the above figure, the incident and reflect powers are given as $P_{\text{inc}} = V_{\text{inc}} I_{\text{inc}}^*$ and $P_{\text{ref}} = |\Gamma_m|^2 P_{\text{inc}}$, respectively.

According to the power conservation principle

$$P_{\text{inc}} (1 - |\Gamma_m|^2) = P_{\text{rad}} + P_{\text{match}} + P_{\text{near}}$$
Matching the real and imaginary components in the above equation gives

\[
\text{Re}(P_{\text{inc}}) \left( 1 - |\Gamma_m|^2 \right) = P_{\text{rad}} \\
\text{Im}(P_{\text{inc}}) \left( 1 - |\Gamma_m|^2 \right) = P_{\text{match}} + P_{\text{near}}
\]

- \( P_{\text{rad}} \Rightarrow \) (real) power radiated by the antenna
- \( P_{\text{near}} \Rightarrow \) (reactive) near-field power of the antenna
- \( P_{\text{match}} \Rightarrow \) (reactive) power in the matching network

From the field perspective

If the incident power is \( P_{\text{inc}}^{\text{ref}} \) (“ref” stands for reference), then the far-field (radiated) electric field is given as

\[
\vec{E}_{\text{EP}}^{\text{far}}(\theta, \phi) = A_{\text{ref}} \frac{\exp\left( -jk_0 r \right)}{r} e^{j\theta}(\theta, \phi).
\]

Similarly, for the same antenna, the electric field radiated in the case of incident power \( P_{\text{inc}}' \) is given by

\[
\vec{E}_{\text{EP}}^{\text{far}}(\theta, \phi) = A' \frac{\exp\left( -jk_0 r \right)}{r} e^{j\theta}(\theta, \phi)
\]

Then the relation between \( A_{\text{ref}} \) and \( A' \) is

\[
\frac{|A'|}{|A_{\text{ref}}|} = \sqrt{\frac{P_{\text{inc}}'}{P_{\text{inc}}^{\text{ref}}}}
\]

because of linearity.

In addition, the relation between \( \angle A_{\text{ref}} \) and \( \angle A' \) is

\[
\frac{\angle A'}{\angle A_{\text{ref}}} = \Phi' - \Phi
\]

where \( \Phi' \) and \( \Phi \) are phase shifts (caused by the variable phase shifter) corresponding to \( P_{\text{inc}}' \) and \( P_{\text{inc}}^{\text{ref}} \), respectively. Therefore, finally, the relation between \( A_{\text{ref}} \) and \( A' \) can written as

\[
A' = A_{\text{ref}} \sqrt{\frac{P_{\text{inc}}'}{P_{\text{inc}}^{\text{ref}}} \exp[j(\Phi' - \Phi)]}.
\]
Now consider the situation of an array of \( N \) number of elements

Far-field of the entire array is given as

\[
\mathbf{E}_{\text{array}} = \mathbf{e}^{\text{far}}(\theta, \phi) \sum_{n=1}^{N} A_n \exp \left( -j k_0 r_n \right) \]

\[
= A^{\text{ref}} \exp \left( -j \Phi_1 \right) \sum_{n=1}^{N} \left\{ \frac{P_{n}^{\text{inc}}}{P_{\text{ref} \text{ inc}}} \exp \left( j \Phi_n \right) \exp \left( -j k_0 r_n \right) \right\} \]

\[
\approx A^{\text{ref}} \exp \left( -j \Phi_1 \right) \sum_{n=1}^{N} \left\{ \frac{P_{n}^{\text{inc}}}{P_{\text{ref} \text{ inc}}} \exp \left[ j \left( \Phi_n + k_0 \cdot \mathbf{r}_n^T \right) \right] \right\} \]

where \( k_0 = k_0 (\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}) \) and \( \mathbf{r}_n^T \) is the position vector corresponding to the \( n^{th} \) element. From the above equation, directivity \( D_{\text{array}} \) is given as

\[
D_{\text{array}} = \frac{4 \pi \left| \mathbf{E}_{\text{array}} \times \mathbf{E}_{\text{array}}^* \right|}{\int_{0}^{2 \pi} \int_{0}^{\pi} \left| \mathbf{E}_{\text{array}} \times \mathbf{E}_{\text{array}}^* \right| \sin \theta d\theta d\phi} = \frac{4 \pi \left( |A F (\theta, \phi) \mathbf{e}^{\text{far}} (\theta, \phi)|^2 \right)}{\int_{0}^{2 \pi} \int_{0}^{\pi} \left( |A F (\theta, \phi) \mathbf{e}^{\text{far}} (\theta, \phi)|^2 \right) \sin \theta d\theta d\phi} \tag{1.1} \]

where \( A F (\theta, \phi) = \sum_{n=1}^{N} \left\{ \sqrt{\frac{P_{n}^{\text{inc}}}{P_{\text{ref} \text{ inc}}}} \exp \left[ j \left( \Phi_n + k_0 \cdot \mathbf{r}_n^T \right) \right] \right\} \).

Similarly, gain of the array \( G_{\text{array}} \) is given as

\[
G_{\text{array}} = \frac{4 \pi \left| \mathbf{E}_{\text{array}} \times \mathbf{E}_{\text{array}}^* \right|}{\eta P_{\text{tot} \text{ inc}}} \]

where \( \eta \) is the free-space impedance. The above equation can be further reduced to

\[
G_{\text{array}} = \frac{4 \pi \left| A^{\text{ref}} \right|^2 \left| \mathbf{e}^{\text{far}} (\theta, \phi) \right|^2 \left| A F (\theta, \phi) \right|^2}{\eta} \sum_{n=1}^{N} (P_{n}^{\text{inc}})^2 \]

\[
= \frac{4 \pi \left| A^{\text{ref}} \right|^2 \left| \mathbf{e}^{\text{far}} (\theta, \phi) \right|^2 \left| \sum_{n=1}^{N} \left\{ \sqrt{\frac{P_{n}^{\text{inc}}}{P_{\text{ref} \text{ inc}}}} \exp \left[ j \left( \Phi_n + k_0 \cdot \mathbf{r}_n^T \right) \right] \right\} \right|^2}{\eta} \sum_{n=1}^{N} (P_{n}^{\text{inc}}) \]

\[
= G (\theta, \phi) \left| \sum_{n=1}^{N} \left\{ \sqrt{\frac{P_{n}^{\text{inc}}}{P_{\text{ref} \text{ inc}}}} \exp \left[ j \left( \Phi_n + k_0 \cdot \mathbf{r}_n^T \right) \right] \right\} \right|^2 \tag{1.2} \]

where \( G (\theta, \phi) \) is the gain of the individual element.
Now, let us consider an IDEAL situation when there is no reactive power in the entire system, then

\[ P_{\text{inc}}^n = P_{\text{rad}}^n. \]

So, in that IDEAL situation,

\[
G_{\text{array}} \text{ or } D_{\text{array}} = D(\theta, \phi) \frac{\sum_{n=1}^{N} \left\{ \sqrt{P_{\text{rad}}^n} \exp \left[j \left( \Phi_n + k_n^0 \cdot i_n \right) \right] \right\}^2}{\sum_{n=1}^{N} (P_{\text{rad}}^n)}
\]

where \( D(\theta, \phi) \) is the directivity of the individual element.

In other words, if we know \( P_{\text{rad}}^n \) somehow, then the directivity of the entire array \( D_{\text{array}} \) is given by the above equation.