Chapter 1

Functional Invariance, Noether’s Theorem, and Principle of Least Action

1.1 Functional Invariance

Definition. A functional of the form

\[ J[y_1, \cdots, y_n] = \int_{x_0}^{x_1} F(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \, dx \]

is said to be invariant under the transformation

\[
\begin{align*}
  x^* & = \Phi(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \\
  y_i^* & = \Psi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n')
\end{align*}
\]

if

\[
\int_{x_0}^{x_1} F(x^*, y_1^*, \cdots, y_n^*, \frac{dy_1^*}{dx^*}, \cdots, \frac{dy_n^*}{dx^*}) \, dx^* = \int_{x_0}^{x_1} F(x, y_1, \cdots, y_n, \frac{dy_1}{dx}, \cdots, \frac{dy_n}{dx}) \, dx
\]

where \( i = 1, \cdots, n \).

1.2 Noether’s Theorem

Suppose now that we have a family of transformations

\[
\begin{align*}
  x^* & = \Phi(x, y_1, \cdots, y_n, y_1', \cdots, y_n'; \varepsilon) \\
  y_i^* & = \Psi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n'; \varepsilon)
\end{align*}
\]

depending on a parameter \( \varepsilon \), where the functions \( \Phi \) and \( \Psi_i \) are differentiable with respect to \( \varepsilon \), and the value \( \varepsilon = 0 \) corresponds to the identity transformation:

\[
\begin{align*}
  \Phi(x, y_1, \cdots, y_n, y_1', \cdots, y_n' ; 0) & = x \\
  \Psi_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n' ; 0) & = y_i
\end{align*}
\]

Then we have the following result:
Theorem. If the functional
\[ J [y_1, \ldots, y_n] = \int_{x_0}^{x_1} F(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n) \, dx \]
is invariant under the family of transformations (1.1) for arbitrary \( x_0 \) and \( x_1 \), then
\[ \sum_{i=1}^{n} F_{y'i} \psi_i + \left( F - \sum_{i=1}^{n} F_{y'i} y'_i \right) \varphi = \sum_{i=1}^{n} p_i \psi_i - H \varphi = \text{const} \] (1.3)
along each extremal of \( J [y_1, \ldots, y_n] \), where
\[
\begin{aligned}
\varphi (x, y_1, \ldots, y_n, y'_1, \ldots, y'_n) &= \frac{\partial \Phi(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n; \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0} \\
\psi_i (x, y_1, \ldots, y_n, y'_1, \ldots, y'_n, 0) &= \frac{\partial \Psi_i(x, y_1, \ldots, y_n, y'_1, \ldots, y'_n; \varepsilon)}{\partial \varepsilon} \bigg|_{\varepsilon=0}.
\end{aligned}
\] (1.4)
In other words, every one-parameter family of transformations leaving \( J [y_1, \ldots, y_n] \) invariant leads to a first integral of its system of Euler-Lagrange equations.

1.3 Principle of Least Action\(^1\)

We now apply the concept of variation to obtain Newton’s equations of motion for a system of \( n \) particles.

Suppose we are given a system of \( n \) particles (mass points), where no constraints whatsoever are imposed on the system. Let \( i \)th particle have mass \( m_i \) and coordinates \( x_i, y_i, z_i \). Then the kinetic energy \( T \) of the system is\(^2\)
\[ T = \frac{1}{2} \sum_{i=1}^{n} m_i \left( \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right). \]

We assume that the system has potential energy \( U \), i.e., that there exist a function \( U = U(t, x_1, y_1, z_1, \ldots, x_n, y_n, z_n) \) such that the force acting on the \( i \)th particle is \( \vec{F}_i = F_{i}^{x} \dot{x} + F_{i}^{y} \dot{y} + F_{i}^{z} \dot{z} \), where
\[ F_{i}^{x} = -\frac{\partial U}{\partial x_i}, \quad F_{i}^{y} = -\frac{\partial U}{\partial y_i}, \quad F_{i}^{z} = -\frac{\partial U}{\partial z_i}. \]

Next, we introduce the expression
\[ L(t, x_1, y_1, z_1, \ldots, x_n, y_n, z_n, \dot{x}_1, \dot{y}_1, \dot{z}_1, \ldots, \dot{x}_n, \dot{y}_n, \dot{z}_n) = T - U \] (1.5)
called Lagrangian (function) of the system of particles.

\(^1\)or, more accurately, the principle of stationary action
\(^2\)Here, \( t \) denotes time, and the overdot denotes differentiation with respect to \( t \).
Theorem. The motion of a system of \(n\) particles during the time interval \([t_0, t_1]\) is described by those functions \(x_i(t), y_i(t), z_i(t), 1 \leq i \leq n\), for which the integral

\[
\int_{t_0}^{t_1} L(t, x_1, y_1, z_1, \cdots, x_n, y_n, z_n, \dot{x}_1, \dot{y}_1, \dot{z}_1, \cdots, \dot{x}_n, \dot{y}_n, \dot{z}_n) \, dt
\]

(1.6)
called the action, is a minimum.

So, from the above theorem and Euler-Lagrange equations, we get

\[
\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = 0,
\]

\[
\frac{\partial L}{\partial y_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{y}_i} = 0,
\]

\[
\frac{\partial L}{\partial z_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{z}_i} = 0.
\]

(1.7)

Substituting (1.5) in the above equations gives

\[
-\frac{\partial U}{\partial x_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{x}_i} = 0,
\]

\[
-\frac{\partial U}{\partial y_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{y}_i} = 0,
\]

\[
-\frac{\partial U}{\partial z_i} - \frac{d}{dt} \frac{\partial T}{\partial \dot{z}_i} = 0.
\]

(1.8)

Further reducing the above equations gives

\[
F_i^x = m_i \ddot{x}_i,
\]

\[
F_i^y = m_i \ddot{y}_i,
\]

\[
F_i^z = m_i \ddot{z}_i.
\]

(1.9)

### 1.4 Conservation Laws

In the previous section we used the concept of variation to obtain the Newton’s equations. In this section we further analyze the functional action and with the help of Noether’s theorem we will prove various conservation laws. First of all, let us write down the canonical variables corresponding to the functional (1.6). They are,

\[
p_{ix} = \frac{\partial L}{\partial \dot{x}_i} = m_i \ddot{x}_i,
\]

\[
p_{iy} = \frac{\partial L}{\partial \dot{y}_i} = m_i \ddot{y}_i,
\]

\[
p_{iz} = \frac{\partial L}{\partial \dot{z}_i} = m_i \ddot{z}_i,
\]

\[
H = -L + \sum_{i=1}^{n} (p_{ix} \dot{x}_i + p_{iy} \dot{y}_i + p_{iz} \dot{z}_i) = U + T.
\]
So, canonical variables are nothing but the components of the momentums of all particles and total energy of the system. Now, let us come back to the Noether’s Theorem and prove various conservation laws in the following subsections.

### 1.4.1 Conservation of Energy

Suppose the given system is *conservative*, which means that the Lagrangian \( L \) does not depend on time explicitly. Then the functional (1.6) is invariant under the transformation

\[
\begin{align*}
  t^* &= t + \varepsilon \\
  x_i^* &= x_i \\
  y_i^* &= y_i \\
  z_i^* &= z_i
\end{align*}
\]

Then from the Noether’s theorem,

\[
\sum_{i=1}^{n} (p_{ix} \psi_{ix} + p_{iy} \psi_{iy} + p_{iz} \psi_{iz}) - H \varphi = \text{const.} \tag{1.10}
\]

However,

\[
\begin{align*}
  \varphi &= \frac{\partial t^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 1 \\
  \psi_{ix} &= \frac{\partial x_i^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 \\
  \vdots &= \vdots &= \vdots
\end{align*}
\]

So, substituting the above equation in (1.10) gives

\[
H = T + U = \text{const} \tag{1.12}
\]

along each extremal.

### 1.4.2 Conservation of Momentum

If the functional (1.6) is invariant under the transformation

\[
\begin{align*}
  t^* &= t \\
  x_i^* &= x_i + \varepsilon \\
  y_i^* &= y_i \\
  z_i^* &= z_i
\end{align*}
\]

Then from the Noether’s theorem,

\[
\sum_{i=1}^{n} (p_{ix} \psi_{ix} + p_{iy} \psi_{iy} + p_{iz} \psi_{iz}) - H \varphi = \text{const.} \tag{1.13}
\]

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However,

\[
\begin{align*}
\varphi &= \frac{\partial t^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 \\
\psi_{ix} &= \frac{\partial x_i^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 1 \\
\psi_{iy} &= \frac{\partial y_i^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 \\
\psi_{iz} &= \frac{\partial z_i^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0.
\end{align*}
\]

(1.14)

So, substituting the above equation in (1.13) gives

\[
\sum_{i=1}^{n} p_{ix} = \text{const.}
\]

(1.15)

Similarly, it follows from the invariance of (1.6) under displacements along the \(y\)-axis that

\[
\sum_{i=1}^{n} p_{iy} = \text{const},
\]

(1.16)

and from the invariance of (1.6) under displacements along the \(z\)-axis that

\[
\sum_{i=1}^{n} p_{iz} = \text{const}.
\]

(1.17)

Thus the total momentum is conserved during the motion of the system if the integral (1.6) is invariant under parallel displacement.

### 1.4.3 Conservation of Angular Momentum

If the functional (1.6) is invariant under the transformation

\[
\begin{align*}
t^* &= t \\
x_i^* &= x_i \cos \varepsilon + y_i \sin \varepsilon \\
y_i^* &= -x_i \sin \varepsilon + y_i \cos \varepsilon \\
z_i^* &= z_i
\end{align*}
\]

Then from the Noether’s theorem,

\[
\sum_{i=1}^{n} (p_{ix} \psi_{ix} + p_{iy} \psi_{iy} + p_{iz} \psi_{iz}) - H \varphi = \text{const.}
\]

(1.18)

However,

\[
\begin{align*}
\varphi &= \frac{\partial t^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0 \\
\psi_{ix} &= \frac{\partial x_i^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = y_i \\
\psi_{iy} &= \frac{\partial y_i^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = -x_i \\
\psi_{iz} &= \frac{\partial z_i^*}{\partial \varepsilon} \bigg|_{\varepsilon=0} = 0
\end{align*}
\]

(1.19)
So, substituting the above equation in (1.18) gives

\[ \sum_{i=1}^{n} (p_{ix}y_i - p_{iy}x_i) = \text{const.} \quad (1.20) \]

Each term in the above sum represents the z-component of the vector product \( \vec{p}_i \times \vec{r}_i \), where \( \vec{p}_i = p_{ix}\hat{x} + p_{iy}\hat{y} + p_{iz}\hat{z} \) and \( \vec{r}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z} \). The vector \( \vec{p}_i \times \vec{r}_i \) is called the angular momentum of the \( i^{th} \) particle, about the origin of coordinates. Thus we have proved that the total angular momentum does not change during the motion of the system if (1.6) is invariant under all rotations.

The material presented here is from the book: