Contents

Chapter 1

Network Parameters of a Two-Port Filter

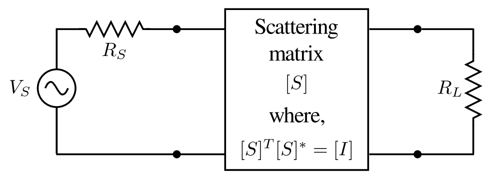


Figure 1.1: S parameter representation of a linear, loss-less and reciprocal (LLR) device.

1.1 S Parameters in the s-domain

For a LLR device, if the scattering parameter S_{11} is given in $s = (\sigma + j\omega)$ domain as[1]

$$S_{11}(s) = \frac{1}{\varepsilon_R} \frac{F(s)}{E(s)}, \tag{1.1}$$

then, from the *unitary* property of scattering matrices of LLR devices,

$$\begin{aligned} S_{21}(s) S_{21}(s)^* = 1 - S_{11}(s) S_{11}(s)^* &= 1 - \frac{1}{|\varepsilon_R|^2} \left| \frac{F(s)}{E(s)} \right|^2 \\ &= \frac{|\varepsilon_R|^2 |E(s)|^2 - |F(s)|^2}{|\varepsilon_R|^2 |E(s)|^2}. \end{aligned} \tag{1.2}$$

If $S_{21}(s)$ can be written as $\frac{1}{\varepsilon} \frac{P(s)}{E(s)}$, then

$$\frac{|P(s)|^2}{|\varepsilon|^2} = \frac{|\varepsilon_R|^2 |E(s)|^2 - |F(s)|^2}{|\varepsilon_R|^2}. \tag{1.3}$$

[1] All polynomials we are dealing in this section are assumed to be normalized (by ε and ε_R) such that their highest degree coefficients are equal to 1.

Also, re-arranging (1.2) and (1.3) gives

$$|S_{21}(s)|^2 = \frac{1}{1 + \left|\frac{\varepsilon}{\varepsilon_R}\right|^2 \left|\frac{F(s)}{P(s)}\right|^2}, \tag{1.4}$$

where $\frac{F(s)}{P(s)}$ is referred to as *characteristic function*. Before proceeding further, some important properties of the polynomials $E(s)$, $F(s)$ and $P(s)$ are given[2] here.

1. $E(s)$ is a Hurwitz polynomial of degree N. All its roots lie in the left half-plane of s.
2. The roots of $F(s)$ lie on the imaginary axis where the degree is N. These roots are known as *reflection zeros*.
3. Zeros of the polynomial $P(s)$ are known as *transmission zeros* (TX zeros). All TX zeros lie on the imaginary axis or appear as pairs of zeros located symmetrically with respect to the imaginary axis.[3]

As a result, the polynomials $E(s)$, $F(s)$ and $P(s)$ have the forms

$$\begin{aligned} E(s) &= s^N + e_{N-1}s^{N-1} + e_{N-2}s^{N-2} + e_{N-3}s^{N-3} + \cdots + e_0, \\ F(s) &= s^N + jf_{N-1}s^{N-1} + f_{N-2}s^{N-2} + jf_{N-3}s^{N-3} + \cdots + f_0 \text{ and} \\ P(s) &= s^{n_{fz}} + jp_{n_{fz}-1}s^{n_{fz}-1} + p_{n_{fz}-2}s^{n_{fz}-2} + jp_{n_{fz}-3}s^{n_{fz}-3} + \cdots + p_0. \end{aligned} \tag{1.5}$$

In the above equations, all the coefficients e_i are complex. Except f_0 and p_0, all other parameters f_i and p_i are real. Since the coefficients of $F(s)$ and $P(s)$ alternate between real and imaginary numbers, f_0 and p_0 are real if N and n_{fz} are even (and imaginary if their orders are odd).

Also, the following statements can be proved without much difficulty.

$$\text{at } s = 0: \begin{cases} S_{11} = & \frac{1}{\varepsilon_R}\frac{f_0}{e_0} \\ S_{21} = & \frac{1}{\varepsilon}\frac{p_0}{e_0} \end{cases} \tag{1.6}$$

$$\text{at } s = \pm j\infty: \begin{cases} S_{11} = & \frac{1}{\varepsilon_R} \\ S_{21} = & 0, \text{ if } n_{fz} < N \\ S_{21} = & \frac{1}{\varepsilon}, \text{ if } n_{fz} = N \end{cases} \tag{1.7}$$

1.1.1 Relation between ε and ε_R

As mentioned before, ε and ε_R are just some *complex* numbers for normalizing all the polynomials. The relation between these two parameters can be derived from the equation

$$\begin{aligned} S_{11}(s)\,S_{11}(s)^* + S_{21}(s)\,S_{21}(s)^* &= 1 \\ \Rightarrow \frac{1}{|\varepsilon_R|^2}\frac{F(s)\,F(s)^*}{E(s)\,E(s)^*} + \frac{1}{|\varepsilon|^2}\frac{P(s)\,P(s)^*}{E(s)\,E(s)^*} &= 1. \end{aligned} \tag{1.8}$$

[2] For the time being, these properties are stated without any proof.

[3] If n_{fz}, the degree of the polynomial $P(s)$ is zero, then all transmission zeros are located at $s = \pm j\infty$ (e.g., conventional Butterworth, Chebyshev, etc.).

At $s = 0$, (1.8) becomes

$$\begin{aligned} S_{11}(s) S_{11}(s)^* + S_{21}(s) S_{21}(s)^* &= 1 \\ \Rightarrow \frac{1}{|\varepsilon_R|^2}\left|\frac{f_0}{e_0}\right|^2 + \frac{1}{|\varepsilon|^2}\left|\frac{p_0}{e_0}\right|^2 &= 1. \end{aligned} \tag{1.9}$$

Similarly, at $s = \pm j\infty$,

$$\begin{cases} |\varepsilon_R| = 1, & \text{if } n_{fz} < N \\ \frac{1}{|\varepsilon_R|^2} + \frac{1}{|\varepsilon|^2} = 1, & \text{if } n_{fz} = N \end{cases}. \tag{1.10}$$

So, from (1.9) and (1.10)

$$\text{if } n_{fz} < N : \begin{cases} |\varepsilon_R| = 1, \\ |\varepsilon|^2 = \frac{|p_0|^2}{|e_0|^2 - |f_0|^2}. \end{cases} \tag{1.11}$$

$$\text{if } n_{fz} = N : \begin{cases} |\varepsilon|^2 = \frac{|f_0|^2 - |p_0|^2}{|f_0|^2 - |e_0|^2}, \\ |\varepsilon_R|^2 = \frac{|f_0|^2 - |p_0|^2}{|e_0|^2 - |p_0|^2}. \end{cases} \tag{1.12}$$

It is important to understand that only $|\varepsilon_R|$ and $|\varepsilon|$ are related to each other, but not $\angle\varepsilon$ and $\angle\varepsilon_R$. The phases can be changed *arbitrarly* by shifting the reference planes of the two ports.

1.1.2 Derivation of S_{22}

Since S_{11}, S_{12} and S_{21} are already known, one can derive S_{22} from the following S-parameter unitary property:

$$\begin{aligned} S_{11}(s) S_{12}(s)^* + S_{21}(s) S_{22}(s)^* &= 0 \\ \Rightarrow S_{22} &= -\left(\frac{\varepsilon^*}{\varepsilon\varepsilon_R^*}\right)\frac{F(s)^*}{E(s)}\frac{P(s)}{P(s)^*}. \end{aligned} \tag{1.13}$$

Further it can be showed that

$$\begin{cases} S_{22} = -\left(\frac{\varepsilon^*}{\varepsilon\varepsilon_R^*}\right)\frac{p_0 f_0^*}{e_0 p_0^*}, & \text{at } s = 0 \\ S_{22} = \left(\frac{\varepsilon^*}{\varepsilon\varepsilon_R^*}\right)(-1)^{N+n_{fz}+1}, & \text{as } s \to \pm j\infty \end{cases}. \tag{1.14}$$

1.2 Network Parameters in the $j\omega$-domain

1.2.1 S Parameters in the $j\omega$-domain

Till now, all the scattering parameters have been dealt in the s domain. Such an analysis provides information regarding the *physical realizability*[4] of the filter. Once the filtering functions are

[4]from the view point of placement of the roots of polynomials $E(s)$, $F(s)$ and $P(s)$

made sure to be physically realizable, one needs to worry about the $j\omega$ domain only. So, S parameters can be written in $j\omega$ domain as

$$\left[\begin{array}{cc} S_{11}(j\omega) & S_{12}(j\omega) \\ S_{21}(j\omega) & S_{22}(j\omega) \end{array}\right] = \left[\begin{array}{cc} \frac{1}{\varepsilon_R}\frac{F(j\omega)}{E(j\omega)} & \frac{1}{\varepsilon}\frac{P(j\omega)}{E(j\omega)} \\ \frac{1}{\varepsilon}\frac{P(j\omega)}{E(j\omega)} & \frac{(-1)^{1+n_{fz}}\varepsilon^*}{\varepsilon\varepsilon_R^*}\frac{F(j\omega)^*}{E(j\omega)} \end{array}\right]. \tag{1.15}$$

1.2.2 $ABCD$ Parameters in the $j\omega$-domain

$ABCD$ parameters of the device can be obtained from (1.15) and are as given below:

$$\left[\begin{array}{cc} A(j\omega) & B(j\omega) \\ C(j\omega) & D(j\omega) \end{array}\right] = \frac{\varepsilon}{2P(j\omega)}\left[\begin{array}{cc} \sqrt{\frac{R_S}{R_L}}\left(\mathbb{EF}_+ + \mathbb{EF}_{+*}\right) & \sqrt{R_S R_L}\left(\mathbb{EF}_+ - \mathbb{EF}_{+*}\right) \\ \frac{1}{\sqrt{R_S R_L}}\left(\mathbb{EF}_- - \mathbb{EF}_{-*}\right) & \sqrt{\frac{R_L}{R_S}}\left(\mathbb{EF}_- + \mathbb{EF}_{-*}\right) \end{array}\right], \tag{1.16}$$

where

$$\begin{aligned} \mathbb{EF}_+ &= \left[E(j\omega) + \frac{F(j\omega)}{\varepsilon_R}\right], \\ \mathbb{EF}_{+*} &= (-1)^{n_{fz}}\left(\frac{\varepsilon^*}{\varepsilon}\right)\left[E(j\omega)^* + \frac{F(j\omega)^*}{\varepsilon_R^*}\right], \\ \mathbb{EF}_- &= \left[E(j\omega) - \frac{F(j\omega)}{\varepsilon_R}\right] \text{ and} \\ \mathbb{EF}_{-*} &= (-1)^{n_{fz}}\left(\frac{\varepsilon^*}{\varepsilon}\right)\left[E(j\omega)^* - \frac{F(j\omega)^*}{\varepsilon_R^*}\right]. \end{aligned}$$

1.2.3 Y Parameters in the $j\omega$-domain

Once $ABCD$ parameters are known, Y parameters (for a reciprocal network) can be easily obtained as shown below:

$$\begin{aligned} \left[\begin{array}{cc} Y_{11}(j\omega) & Y_{12}(j\omega) \\ Y_{21}(j\omega) & Y_{22}(j\omega) \end{array}\right] &= \frac{1}{B(j\omega)}\left[\begin{array}{cc} D(j\omega) & -1 \\ -1 & A(j\omega) \end{array}\right] \\ &= \frac{1}{\left(\mathbb{EF}_+ - \mathbb{EF}_{+*}\right)}\left[\begin{array}{cc} \frac{1}{R_S}\left(\mathbb{EF}_- + \mathbb{EF}_{-*}\right) & -\frac{1}{\sqrt{R_S R_L}}\frac{2P}{\varepsilon} \\ -\frac{1}{\sqrt{R_S R_L}}\frac{2P}{\varepsilon} & \frac{1}{R_L}\left(\mathbb{EF}_+ + \mathbb{EF}_{+*}\right) \end{array}\right] \end{aligned} \tag{1.17}$$

1.2.4 Z Parameters in the $j\omega$-domain

Similarly, Z parameters also can be obtained from the $ABCD$ parameters as shown below:

$$\begin{aligned} \left[\begin{array}{cc} Z_{11}(j\omega) & Z_{12}(j\omega) \\ Z_{21}(j\omega) & Z_{22}(j\omega) \end{array}\right] &= \frac{1}{C(j\omega)}\left[\begin{array}{cc} A(j\omega) & 1 \\ 1 & D(j\omega) \end{array}\right] \\ &= \frac{1}{\left(\mathbb{EF}_- - \mathbb{EF}_{-*}\right)}\left[\begin{array}{cc} R_S\left(\mathbb{EF}_+ + \mathbb{EF}_{+*}\right) & \sqrt{R_S R_L}\frac{2P}{\varepsilon} \\ \sqrt{R_S R_L}\frac{2P}{\varepsilon} & R_L\left(\mathbb{EF}_- + \mathbb{EF}_{-*}\right) \end{array}\right] \end{aligned} \tag{1.18}$$

Chapter 2

Lowpass Prototype Filters

2.1 Basic Components of a Lowpass Prototype Filter

It is customary in the filter design to synthesize the lowpass prototype (LPP) first. From the designed LPP, components of the actual filter can be obtained by using frequency transformations. Two general LPP configurations are shown in Fig. 2.1 and 2.2. In these figures, the colored components are assumed to be frequency invariant (i.e., do not change with frequency transformations). All the other components change according to the actual filter response required (such as bandpass, bandstop, etc) and their transformed values are shown in Fig. 2.4. Also, characteristics of the immittance inverters (both K & J) used in the LPPs are shown in Fig. 2.5. For impedance and admittance inverters, $Z_{\text{in}} = \frac{K^2}{Z_L}$ and $Y_{\text{in}} = \frac{J^2}{Y_L}$, respectively.

2.2 Electric and Magnetic Couplings[1]

The concept of immittance inverters has been mentioned briefly in the previous section. One can physically realize immittance inverters in several ways. Out of all the possible ways, two methods namely electric and magnetic coupling methods are very important in filter designing. These two coupling phenomenas are described in Fig. 2.6 and 2.7. KVL and KCL equations related to both electric as well as magnetic coupling are as given below:

$$\begin{aligned} \text{Magnetic coupling}: \left(j\omega L + \frac{1}{j\omega C}\right) i_1 + jKi_2 = 0,\ \text{where}\ K = -\omega L_m \\ \text{Electric coupling}: \left(j\omega C + \frac{1}{j\omega L}\right) v_1 + jJv_2 = 0,\ \text{where}\ J = -\omega C_m \end{aligned} \tag{2.1}$$

In Fig. 2.6 and 2.7, if each resonator is isolated from the other, then their resonant frequencies are equal $\left(f_0 = \frac{1}{2\pi\sqrt{LC}}\right)$. However, when these two resonators are brought closer to each other, coupling between them yields two distinct resonant frequencies, usually known as f_{even} and f_{odd} (see Table 2.1).

[1] All the theory given in this section is related to synchronous coupling (i.e., the two isolated resonators resonate at the same frequency). For mixed and asynchronous couplings, see [J. S. Hong].

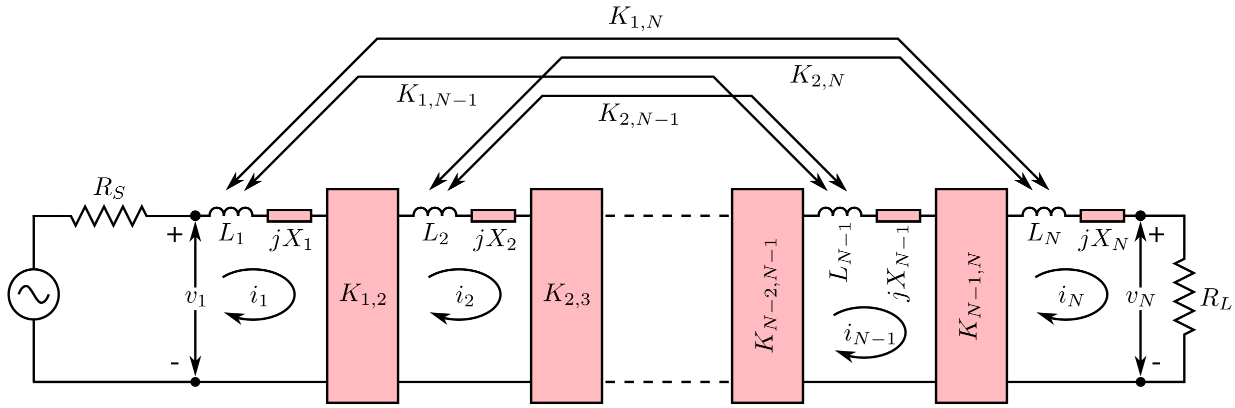


Figure 2.1: A series type LPP

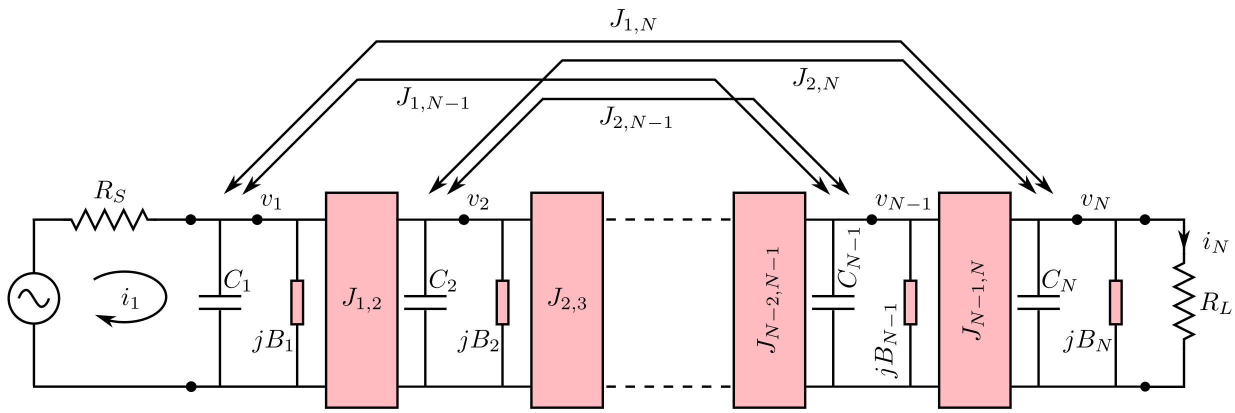


Figure 2.2: A shunt type LPP

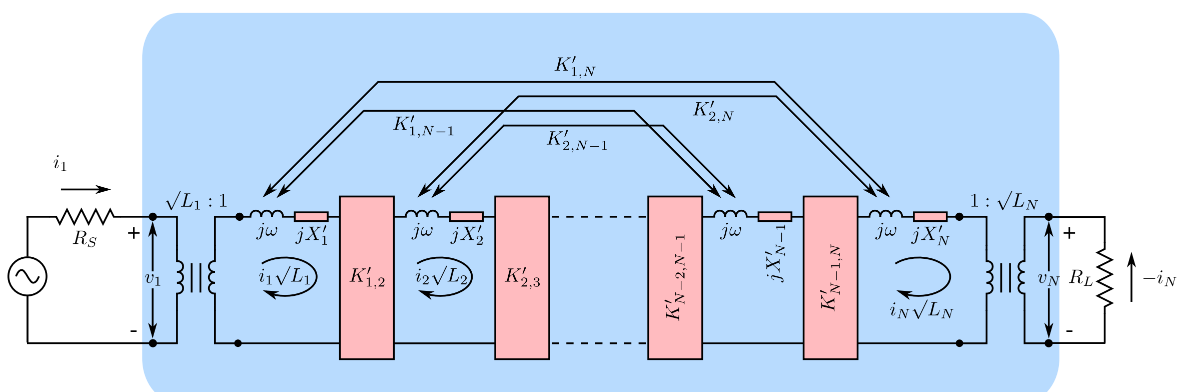


Figure 2.3: Normalized, equivalent circuit of Fig. 2.1.

2.3 ε and ε_R Values for the Prototype Filters

In chapter 1, the relationship between the magnitudes of ε and ε_R was given (1.10). In addition, it was said that for a *general* two-port device, no such relation can be derived for the phases of ε and ε_R. However, if the device under consideration is restricted to be one of the LPP configurations considered in this chapter (e.g., Fig. 2.1 and 2.2), then a more relaxed relation exists between these two parameters.

For example, consider the LPP configuration shown in Fig. 2.1. As $s \to \pm\infty$, it can be showed that both S_{11} and S_{22} tends to the value 1. Therefore, from (1.7) and (1.14),

$$\begin{cases} \varepsilon_R = 1, \text{ and} \\ \frac{\varepsilon}{\varepsilon^*} = (-1)^{\left(N+n_{fz}+1\right)} \end{cases}. \tag{2.2}$$

From the above equation, it can be seen that

$$\begin{cases} \varepsilon = \pm\varepsilon^{\text{re}}, & \text{when } (N+n_{fz}+1) \text{ is even} \\ \varepsilon = \pm j\varepsilon^{\text{re}}, & \text{when } (N+n_{fz}+1) \text{ is odd} \end{cases} \tag{2.3}$$

where ε^{re} is some real number (when $(N+n_{fz}+1)$ is odd, $(N-n_{fz})$ is even).

Similarly, for the LPP configuration shown in Fig. 2.2, ε and ε_R values are given as

$$\begin{cases} \varepsilon_R = -1 \\ \varepsilon = \pm\varepsilon^{\text{re}}, & \text{when } (N+n_{fz}+1) \text{ is even} \\ \varepsilon = \pm j\varepsilon^{\text{re}}, & \text{when } (N+n_{fz}+1) \text{ is odd} \end{cases}. \tag{2.4}$$

Similar results can be obtained for *fully canonical* filter configurations.

2.4 Alternating Pole Method for Determination of $E(j\omega)$

Now that ε, ε_R, $F(j\omega)$ and $P(j\omega)$ are known[2], one needs to evaluate $E(j\omega)$. Re-writing (1.8) in the $j\omega$ domain gives

$$\begin{gathered} |\varepsilon|^2 |\varepsilon_R|^2 E(j\omega) E(j\omega)^* = |\varepsilon_R P(j\omega)|^2 + |\varepsilon F(j\omega)|^2 = \\ \left[\varepsilon_R P(j\omega) + \varepsilon F(j\omega)\right]\left[\varepsilon_R^* P(j\omega)^* + \varepsilon^* F(j\omega)^*\right] - \varepsilon^*\varepsilon_R^*\left[\frac{\varepsilon_R}{\varepsilon_R^*} P(j\omega) F(j\omega)^* + \frac{\varepsilon}{\varepsilon^*} F(j\omega) P(j\omega)^*\right]. \end{gathered} \tag{2.5}$$

For the LPP configurations shown in Fig. 2.1 and 2.2, ε_R is a real number and $\frac{\varepsilon}{\varepsilon^*} = (-1)^{\left(N+n_{fz}+1\right)}$. So, the second term on the right hand side of the above equation becomes

$$\begin{aligned} &\varepsilon^*\varepsilon_R^*\left[\frac{\varepsilon_R}{\varepsilon_R^*} P(j\omega) F(j\omega)^* + \frac{\varepsilon}{\varepsilon^*} F(j\omega) P(j\omega)^*\right] \\ &= \varepsilon^*\varepsilon_R^* P(j\omega) F(j\omega)^* \left[1 + (-1)^{\left(N+n_{fz}+1\right)} \frac{F(j\omega)}{F(j\omega)^*}\frac{P(j\omega)^*}{P(j\omega)}\right] \\ &= \varepsilon^*\varepsilon_R^* P(j\omega) F(j\omega)^* \left[1 + (-1)^{\left(N+n_{fz}+1\right)} (-1)^{\left(N+n_{fz}\right)}\right] \\ &= 0. \end{aligned} \tag{2.6}$$

[2] $F(j\omega)$ and $P(j\omega)$ are given from the desired filter responce.

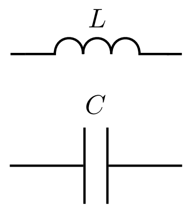
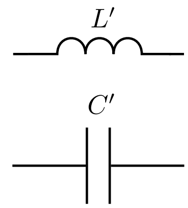
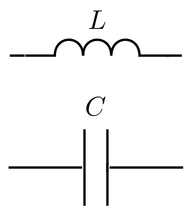
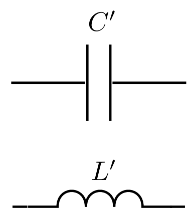
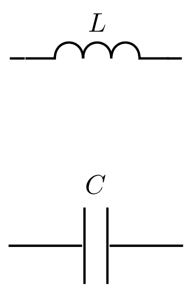
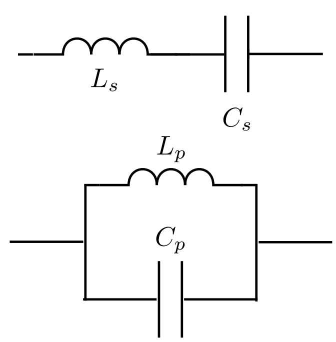
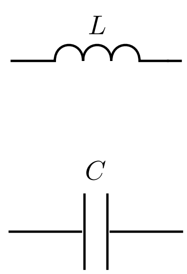
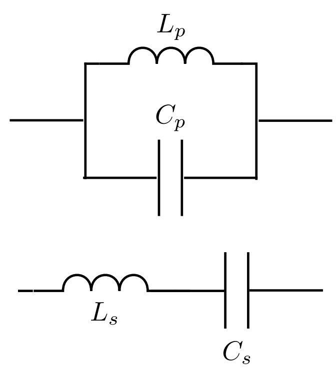
Transformation	LPP element	After Transformation	
Lowpass $\Omega = \frac{\Omega_c \omega}{\omega_c}$			$L' = \frac{\Omega_c}{\omega_c} L$ $C' = \frac{\Omega_c}{\omega_c} C$
Highpass $\Omega = -\frac{\omega_c \Omega_c}{\omega}$			$C' = \frac{1}{\omega_c \Omega_c} \frac{1}{L}$ $L' = \frac{1}{\omega_c \Omega_c} \frac{1}{C}$
Bandpass $\Omega = \frac{\Omega_c}{\mathrm{FBW}} \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)$			$L_s = \frac{\Omega_c}{\mathrm{FBW}\omega_0} L$ $C_s = \frac{1}{\omega_0^2 L_s}$ $C_p = \frac{\Omega_c}{\mathrm{FBW}\omega_0} C$ $L_p = \frac{1}{\omega_0^2 C_p}$
Bandstop $\Omega = \frac{\Omega_c \mathrm{FBW}}{\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)}$			$L_p = \frac{\mathrm{FBW}\Omega_c}{\omega_0} L$ $C_p = \frac{1}{\omega_0^2 L_p}$ $C_s = \frac{\mathrm{FBW}\Omega_c}{\omega_0} C$ $L_s = \frac{1}{\omega_0^2 C_s}$
Ω, ω represent normalized and unnormalized frequency domains, respectievely. $\omega_0 = \sqrt{\omega_1 \omega_2}$ $\mathrm{FBW} = \frac{\omega_2 - \omega_1}{\omega_0}$			

Figure 2.4: Frequency transformation from LPP to bandpass, bandstop, etc.

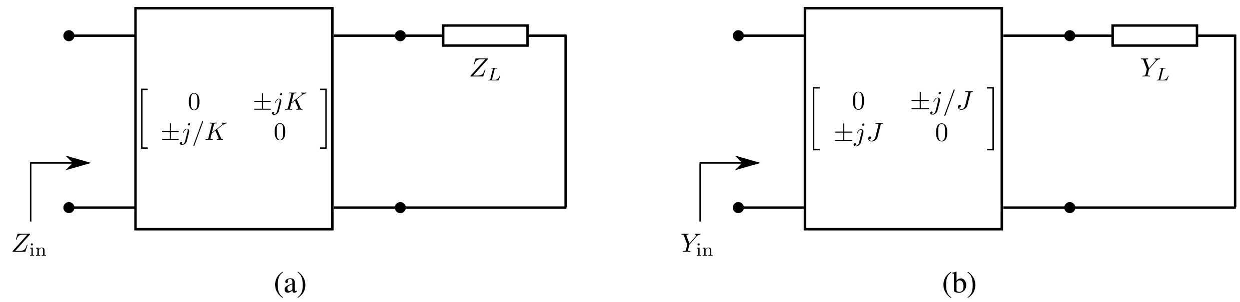


Figure 2.5: Immittance inverters: (a) Impedance inverter and (b) Admittance inverter

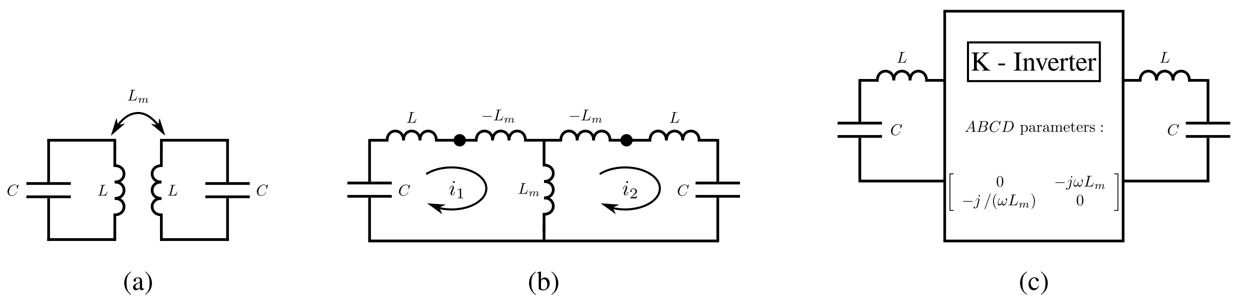


Figure 2.6: Magnetic coupling; (a), (b) and (c) are equivalent.

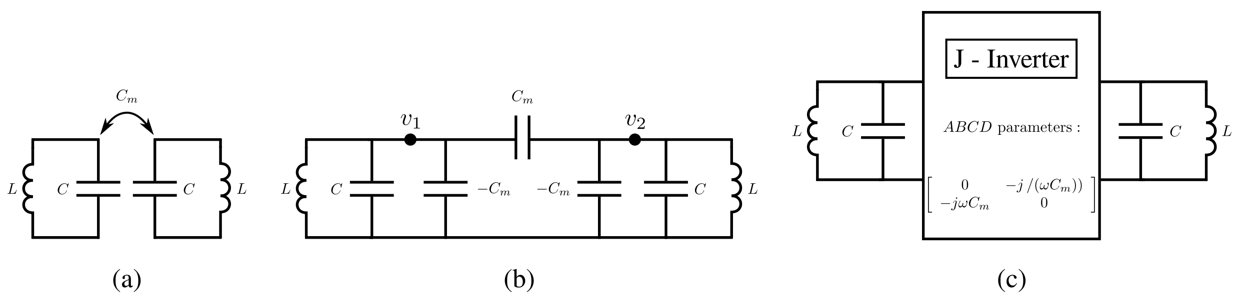


Figure 2.7: Electric coupling; (a), (b) and (c) are equivalent.

	Electric Coupling	Magnetic Coupling
f_{even}	$\frac{1}{2\pi\sqrt{C(L+L_m)}}$	$\frac{1}{2\pi\sqrt{C(C-C_m)}}$
f_{odd}	$\frac{1}{2\pi\sqrt{C(L-L_m)}}$	$\frac{1}{2\pi\sqrt{C(C+C_m)}}$
$\frac{L_m}{L}$ or $\frac{C_m}{C}$	$\frac{f_{\text{odd}}^2-f_{\text{even}}^2}{f_{\text{odd}}^2+f_{\text{even}}^2}$	$\frac{f_{\text{even}}^2-f_{\text{odd}}^2}{f_{\text{even}}^2+f_{\text{odd}}^2}$

Table 2.1: Equations related to electric and magnetic couplings

So,

$$|\varepsilon|^2 |\varepsilon_R|^2 E(j\omega) E(j\omega)^* = [\varepsilon_R P(j\omega) + \varepsilon F(j\omega)] [\varepsilon_R^* P(j\omega)^* + \varepsilon^* F(j\omega)^*] . \quad (2.7)$$

On the imaginary axis (i.e., $s = j\omega$), (2.7) is equivalent to

$$|\varepsilon|^2 |\varepsilon_R|^2 E(s) E(s)^* = [\varepsilon_R P(s) + \varepsilon F(s)] [\varepsilon_R^* P(s)^* + \varepsilon^* F(s)^*] . \quad (2.8)$$

Rooting (in s domain) one of the two terms on the RHS of (2.8) results in a pattern of singularities alternating between left-half and right-half planes. Also, rooting the other term will give the complementary set of singularities. So, it is sufficient to find roots of only one term and then reflect the right-half plane zeros to the left side.

2.5 The N Coupling Matrix

2.5.1 Analysis of the General N Coupling Matrix

2.5.1.1 Series Type LPP

KVL equations corresponding to Fig. 2.1 are given in matrix form as

$$\begin{bmatrix} v_1 \\ 0 \\ \vdots \\ -v_N \end{bmatrix} = j \underbrace{\begin{bmatrix} \omega L_1 + X_1 & K_{1,2} & \cdots & K_{1,N} \\ K_{1,2} & \omega L_2 + X_2 & \cdots & K_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ K_{1,N} & K_{2,N} & \cdots & \omega L_N + X_N \end{bmatrix}}_{[Z]} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} . \quad (2.9)$$

After multiplying the first row by $\frac{1}{\sqrt{L_1}}$, the above matrix representation becomes

$$\begin{bmatrix} \frac{v_1}{\sqrt{L_1}} \\ 0 \\ \vdots \\ -v_N \end{bmatrix} = j \begin{bmatrix} \omega\sqrt{L_1} + \frac{X_1}{\sqrt{L_1}} & \frac{K_{1,2}}{\sqrt{L_1}} & \cdots & \frac{K_{1,N}}{\sqrt{L_1}} \\ K_{1,2} & \omega L_2 + jX_2 & \cdots & K_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ K_{1,N} & K_{2,N} & \cdots & \omega L_N + X_N \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_N \end{bmatrix} .$$

Now, multiplying the first column by $\frac{1}{\sqrt{L_1}}$ gives

$$\begin{bmatrix} \frac{v_1}{\sqrt{L_1}} \\ 0 \\ \vdots \\ -v_N \end{bmatrix} = j \begin{bmatrix} \omega + \frac{X_1}{L_1} & \frac{K_{1,2}}{\sqrt{L_1}} & \cdots & \frac{K_{1,N}}{\sqrt{L_1}} \\ \frac{K_{1,2}}{\sqrt{L_1}} & \omega L_2 + X_2 & \cdots & K_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{K_{1,N}}{\sqrt{L_1}} & K_{2,N} & \cdots & \omega L_N + X_N \end{bmatrix} \begin{bmatrix} i_1\sqrt{L_1} \\ i_2 \\ \vdots \\ i_N \end{bmatrix} .$$

Thus all elements in the above matrix are normalized with respect to L_1. After several similar steps, the final normalized matrix representation is given as

$$\begin{bmatrix} \frac{v_1}{\sqrt{L_1}} \\ 0 \\ \vdots \\ \frac{-v_N}{\sqrt{L_N}} \end{bmatrix} = j \begin{bmatrix} \omega + \frac{X_1}{L_1} & \frac{K_{1,2}}{\sqrt{L_1 L_2}} & \cdots & \frac{K_{1,N}}{\sqrt{L_1 L_N}} \\ \frac{K_{1,2}}{\sqrt{L_1 L_2}} & \omega + \frac{X_2}{L_2} & \cdots & \frac{K_{2,N}}{\sqrt{L_2 L_N}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{K_{1,N}}{\sqrt{L_1 L_N}} & \frac{K_{2,N}}{\sqrt{L_2 L_N}} & \cdots & \omega + \frac{X_N}{L_N} \end{bmatrix} \begin{bmatrix} i_1\sqrt{L_1} \\ i_2\sqrt{L_2} \\ \vdots \\ i_N\sqrt{L_N} \end{bmatrix}$$

$$= j \underbrace{\begin{bmatrix} \omega + X' & K'_{1,2} & \cdots & K'_{1,N} \\ K'_{1,2} & \omega + X'_2 & \cdots & K'_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ K'_{1,N} & K'_{2,N} & \cdots & \omega + X'_N \end{bmatrix}}_{[\mathrm{Z^{norm}}]=j[[\mathrm{M}]+\omega[\mathrm{I}]]} \begin{bmatrix} i_1\sqrt{L_1} \\ i_2\sqrt{L_2} \\ \vdots \\ i_N\sqrt{L_N} \end{bmatrix}, \tag{2.10}$$

where X', $K'_{1,2}$, etc are the normalized values. From (2.9) and (2.10), it is evident that Fig. 2.1 and 2.3 are equivalent. Re-writing (2.10) gives

$$\begin{bmatrix} i_1\sqrt{L_1} \\ i_2\sqrt{L_2} \\ \vdots \\ i_N\sqrt{L_N} \end{bmatrix} = [\mathrm{Z^{norm}}]^{-1} \begin{bmatrix} \frac{v_1}{\sqrt{L_1}} \\ 0 \\ \vdots \\ \frac{-v_N}{\sqrt{L_N}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} i_1\sqrt{L_1} \\ i_N\sqrt{L_N} \end{bmatrix} = \begin{bmatrix} [\mathrm{Z^{norm}}]^{-1}_{11} & [\mathrm{Z^{norm}}]^{-1}_{1N} \\ [\mathrm{Z^{norm}}]^{-1}_{N1} & [\mathrm{Z^{norm}}]^{-1}_{NN} \end{bmatrix} \begin{bmatrix} \frac{v_1}{\sqrt{L_1}} \\ \frac{-v_N}{\sqrt{L_N}} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} i_1 \\ -i_N \end{bmatrix} = \begin{bmatrix} \frac{[\mathrm{Z^{norm}}]^{-1}_{11}}{L_1} & -\frac{[\mathrm{Z^{norm}}]^{-1}_{1N}}{\sqrt{L_1 L_N}} \\ -\frac{[\mathrm{Z^{norm}}]^{-1}_{N1}}{\sqrt{L_1 L_N}} & \frac{[\mathrm{Z^{norm}}]^{-1}_{NN}}{L_N} \end{bmatrix} \begin{bmatrix} v_1 \\ v_N \end{bmatrix}. \tag{2.11}$$

2.5.1.2 Shunt Type LPP

KCL equations corresponding to Fig. 2.2 are given in matrix form as

$$\begin{bmatrix} i_1 \\ 0 \\ \vdots \\ -i_N \end{bmatrix} = j \underbrace{\begin{bmatrix} \omega C_1 + B_1 & J_{1,2} & \cdots & J_{1,N} \\ J_{1,2} & \omega C_2 + B_2 & \cdots & J_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ J_{1,N} & J_{2,N} & \cdots & \omega C_N + B_N \end{bmatrix}}_{[\mathrm{Y}]} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}. \tag{2.12}$$

Normalizing the above matrix gives

$$\begin{aligned}\begin{bmatrix} \frac{i_1}{\sqrt{C_1}} \\ 0 \\ \vdots \\ \frac{-i_N}{\sqrt{C_N}} \end{bmatrix} &= j \begin{bmatrix} \omega + \frac{B_1}{C_1} & \frac{J_{1,2}}{\sqrt{C_1 C_2}} & \cdots & \frac{J_{1,N}}{\sqrt{C_1 C_N}} \\ \frac{J_{1,2}}{\sqrt{C_1 C_2}} & \omega + \frac{B_2}{C_2} & \cdots & \frac{J_{2,N}}{\sqrt{C_2 C_N}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{J_{1,N}}{\sqrt{C_1 C_N}} & \frac{J_{2,N}}{\sqrt{C_2 C_N}} & \cdots & \omega + \frac{B_N}{C_N} \end{bmatrix} \begin{bmatrix} v_1\sqrt{C_1} \\ v_2\sqrt{C_2} \\ \vdots \\ v_N\sqrt{C_N} \end{bmatrix} \\ &= \underbrace{j \begin{bmatrix} \omega + B' & J'_{1,2} & \cdots & J'_{1,N} \\ J'_{1,2} & \omega + B'_2 & \cdots & J'_{2,N} \\ \vdots & \vdots & \vdots & \vdots \\ J'_{1,N} & J'_{2,N} & \cdots & \omega + B'_N \end{bmatrix}}_{[\mathrm{Y}^{\mathrm{norm}}] = j[[\mathrm{M}] + \omega[\mathrm{I}]]} \begin{bmatrix} v_1\sqrt{C_1} \\ v_2\sqrt{C_2} \\ \vdots \\ v_N\sqrt{C_N} \end{bmatrix}, \end{aligned} \tag{2.13}$$

where B', $J'_{1,2}$, etc are the normalized values. Re-writing (2.13) gives

$$\begin{aligned}\begin{bmatrix} v_1\sqrt{C_1} \\ v_2\sqrt{C_2} \\ \vdots \\ v_N\sqrt{C_N} \end{bmatrix} &= [\mathrm{Y}^{\mathrm{norm}}]^{-1} \begin{bmatrix} \frac{i_1}{\sqrt{C_1}} \\ 0 \\ \vdots \\ \frac{-i_N}{\sqrt{C_N}} \end{bmatrix} \\ \Rightarrow \begin{bmatrix} v_1\sqrt{C_1} \\ v_N\sqrt{C_N} \end{bmatrix} &= \begin{bmatrix} [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{11} & [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{1N} \\ [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{N1} & [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{NN} \end{bmatrix} \begin{bmatrix} \frac{i_1}{\sqrt{C_1}} \\ \frac{-i_N}{\sqrt{C_N}} \end{bmatrix} \\ \Rightarrow \begin{bmatrix} v_1 \\ v_N \end{bmatrix} &= \begin{bmatrix} \frac{[\mathrm{Y}^{\mathrm{norm}}]^{-1}_{11}}{C_1} & \frac{[\mathrm{Y}^{\mathrm{norm}}]^{-1}_{1N}}{\sqrt{C_1 C_N}} \\ \frac{[\mathrm{Y}^{\mathrm{norm}}]^{-1}_{N1}}{\sqrt{C_1 C_N}} & \frac{[\mathrm{Y}^{\mathrm{norm}}]^{-1}_{NN}}{C_N} \end{bmatrix} \begin{bmatrix} i_1 \\ -i_N \end{bmatrix}. \end{aligned} \tag{2.14}$$

2.5.2 Synthesis of the General N Coupling Matrix

2.5.2.1 Series Type LPP

From (2.11),

$$\begin{aligned} [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{11} &= L_1 y_{11} \\ \Rightarrow [[\mathrm{M}] + \omega\,[\mathrm{I}]]^{-1}_{11} &= j L_1 y_{11}. \end{aligned} \tag{2.15}$$

Since, $[\mathrm{M}]$ is a real and reciprocal matrix, all of its eigenvalues are real. So, using the eigenvalue decomposition, the above equation can be written as

$$\left[[\mathrm{T}]\,[\Lambda]\,[\mathrm{T}]^t + \omega\,[\mathrm{I}]\right]^{-1}_{11} = j L_1 y_{11}, \tag{2.16}$$

where $[\Lambda] = \mathrm{diag}\,[\lambda_1, \lambda_2, \cdots, \lambda_N]$, λ_i are the eigenvalues of $[\mathrm{M}]$ and $[\mathrm{T}]$ is an orthogonal matrix (i.e., $[\mathrm{T}]\,[\mathrm{T}]^t = [\mathrm{I}]$). In addition, the columns of $[\mathrm{T}]$ are eigenvectors of $[\mathrm{M}]$. The general solution for $(i,j)^{\text{th}}$ element of the left-hand side matrix of (2.16) is given as

$$\begin{aligned}
\left[[\mathrm{T}]\,[\Lambda]\,[\mathrm{T}]^t + \omega\,[\mathrm{I}]\right]_{ij}^{-1} &= \left[[\mathrm{T}]\,[\Lambda]\,[\mathrm{T}]^t + [\mathrm{I}]\,\omega\,[\mathrm{I}]\right]_{ij}^{-1} \\
&= \left[[\mathrm{T}]\,[\Lambda]\,[\mathrm{T}]^t + [\mathrm{T}]\,[\mathrm{T}]^t\,\omega\,[\mathrm{T}]\,[\mathrm{T}]^t\right]_{ij}^{-1} \\
&= \left[[\mathrm{T}]\left[[\Lambda] + [\mathrm{T}]^t\,\omega\,[\mathrm{T}]\right][\mathrm{T}]^t\right]_{ij}^{-1} \\
&= \left[[\mathrm{T}]\left[[\Lambda] + \omega\,[\mathrm{I}]\right][\mathrm{T}]^t\right]_{ij}^{-1} \\
&= \left[[\mathrm{T}]\left[[\Lambda] + \omega\,[\mathrm{I}]\right]^{-1}[\mathrm{T}]^t\right]_{ij} \\
&= \sum_{k=1}^{N} \frac{T_{ik}T_{jk}}{\omega + \lambda_k},\ i,j = 1,2,\cdots,N. \qquad (2.17)
\end{aligned}$$

Therefore from (2.17) and (1.17),

$$\begin{aligned}
\sum_{k=1}^{N} \frac{T_{1k}^2}{\omega + \lambda_k} &= jL_1 y_{11} \\
\Rightarrow \sum_{k=1}^{N} \frac{T_{1k}^2}{\omega + \lambda_k} &= \frac{jL_1}{R_S}\frac{(\mathbb{EF}_- + \mathbb{EF}_{-*})}{(\mathbb{EF}_+ - \mathbb{EF}_{+*})}. \qquad (2.18)
\end{aligned}$$

In the above equation, it can be observed that the numerator is always one degree less than the denominator (i.e., there wont be any constant term when (2.18) is expanded as partial fractions). Similarly, from (2.11),

$$\begin{aligned}
-\frac{[\mathrm{Z}^{\text{norm}}]_{1N}^{-1}}{\sqrt{L_1 L_N}} &= y_{21} \\
\Rightarrow [[\mathrm{M}] + \omega\,[\mathrm{I}]]_{1N}^{-1} &= -j\sqrt{L_1 L_N}\,y_{21} \\
\Rightarrow [[\mathrm{T}]\,[\Lambda]\,[\mathrm{T}] + \omega\,[\mathrm{I}]]_{1N}^{-1} &= -j\sqrt{L_1 L_N}\,y_{21} \\
\Rightarrow \sum_{k=1}^{N} \frac{T_{1k}T_{Nk}}{\omega + \lambda_k} &= j\sqrt{\frac{L_1 L_N}{R_S R_L}}\frac{2P}{\varepsilon\,(\mathbb{EF}_+ - \mathbb{EF}_{+*})}. \qquad (2.19)
\end{aligned}$$

From (2.18) and (2.19) it can be said that $-\lambda_k$ values are zeros of $(\mathbb{EF}_+ - \mathbb{EF}_{+*})$, and $T_{1k} \& T_{Nk}$ values are *related* to the residues at those zeros.

Determination of $\frac{R_S}{L_1}$ and $\frac{R_L}{L_N}$

So far, the known parameters are $\varepsilon, \varepsilon_R, F(j\omega), P(j\omega)$ and $E(j\omega)$. In addition, from the properties of orthogonal matrices,

$$\begin{cases} \sum_{k=1}^{N} T_{1k}^2 &= 1 \\ \sum_{k=1}^{N} T_{Nk}^2 &= 1 \end{cases}. \qquad (2.20)$$

If it is assumed that

$$\sum_{k=1}^{N} \frac{G_{1k}^2}{\omega + \lambda_k} = j \frac{(\mathbb{EF}_- + \mathbb{EF}_{-*})}{(\mathbb{EF}_+ - \mathbb{EF}_{+*})},$$

then from (2.18),

$$T_{1k} = G_{1k} \sqrt{\frac{L_1}{R_S}}. \tag{2.21}$$

Combining (2.21) and (2.20) gives[3]

$$\begin{aligned} \sum_{k=1}^{N} \frac{L_1}{R_S} G_{1k}^2 &= 1 \\ \Rightarrow \frac{R_S}{L_1} &= \sum_{k=1}^{N} G_{1k}^2. \end{aligned} \tag{2.22}$$

Similarly, it can be showed that

$$\begin{aligned} T_{Nk} &= G_{Nk} \sqrt{\frac{L_N}{R_L}}, \text{ where} \\ \frac{R_L}{L_N} &= \sum_{k=1}^{N} G_{Nk}^2. \end{aligned} \tag{2.23}$$

So, for a given filter response, one can determine the matrix $[\Lambda]$ and 1^{st} and N^{th} rows of the matrix $[\mathrm{T}]$. If the matrix $[\mathrm{T}]$ is a 3^{rd} order matrix, then the remaining row (i.e., the second row) can be easily determined. No such simple *unique* solution exists if the order of the filter is greater than 3. So, usually those remaining rows are found by using the Gram-Schmidt orthonormalization process with starting independent vectors[4] as $(T_{11}, T_{12}, \cdots, T_{1N})$, $(T_{N1}, T_{N2}, \cdots, T_{NN})$, $(0, 0, 1, \cdots, 0)$,$\cdots$ and $(0, 0, 0, \cdots, 1)$,. All these vectors are independent as long as $T_{11} \neq 0$ and $T_{N2} \neq 0$. Otherwise, a different set of vectors should be chosen.

[3]It is the ration between R_S and L_1 that is important, not their actual values. So, without loss of generality, many authors simply assume that $L_1 = 1\mathrm{H}$.

[4]Except for the first and last, all the other *independent* vectors are chosen here (kind of) randomly. One can choose any other combination of vectors if he/she wants!

2.5.2.2 Shunt Type LPP

From (2.14) and (1.18),

$$\begin{aligned}
\left[\mathrm{Y}^{\mathrm{norm}}\right]_{11}^{-1} &= C_1 z_{11} \\
\Rightarrow \left[[\mathrm{M}] + \omega\,[\mathrm{I}]\right]_{11}^{-1} &= jC_1 z_{11} \\
\Rightarrow \left[[\mathrm{T}]\,[\Lambda]\,[\mathrm{T}]^t + \omega\,[\mathrm{I}]\right]_{11}^{-1} &= jC_1 z_{11} \\
\Rightarrow \sum_{k=1}^{N} \frac{T_{1k}^2}{\omega + \lambda_k} &= jC_1 z_{11} \\
\Rightarrow \sum_{k=1}^{N} \frac{T_{1k}^2}{\omega + \lambda_k} &= jR_S C_1 \frac{(\mathbb{EF}_+ + \mathbb{EF}_{+*})}{(\mathbb{EF}_- - \mathbb{EF}_{-*})}. \qquad (2.24)
\end{aligned}$$

Once again, $[\Lambda] = \operatorname{diag}[\lambda_1, \lambda_2, \cdots, \lambda_N]$, λ_i are the eigenvalues of $[\mathrm{M}]$, and $[\mathrm{T}]$ is an orthogonal matrix. Similarly, from (2.14) and (1.18),

$$\begin{aligned}
\left[\mathrm{Y}^{\mathrm{norm}}\right]_{1N}^{-1} &= \sqrt{C_1 C_N} z_{21} \\
\Rightarrow \left[[\mathrm{M}] + \omega\,[\mathrm{I}]\right]_{1N}^{-1} &= j\sqrt{C_1 C_N} z_{21} \\
\Rightarrow \left[[\mathrm{T}]\,[\Lambda]\,[\mathrm{T}] + \omega\,[\mathrm{I}]\right]_{1N}^{-1} &= j\sqrt{C_1 C_N} z_{21} \\
\Rightarrow \sum_{k=1}^{N} \frac{T_{1k} T_{Nk}}{\omega + \lambda_k} &= j\sqrt{R_S R_L C_1 C_N} \frac{2P}{\varepsilon\,(\mathbb{EF}_- - \mathbb{EF}_{-*})}. \qquad (2.25)
\end{aligned}$$

2.6 The $N + 2$ Coupling Matrix

Till now, it is assumed that coupling is an intra resonator phenomena. In addition, if coupling between source/load to inner resonators is allowed, then fully canonical filter responses (i.e., $n_{fz} = N$) too can be achieved. A LPP configuration with source/load to inner resonator couplings is shown in Fig. 2.8.

2.6.1 Analysis of the General $N + 2$ Coupling Matrix

2.6.1.1 Series Type LPP

KVL equations corresponding to Fig. 2.8 are given in matrix form as

$$\begin{bmatrix} v_S \\ 0 \\ 0 \\ \vdots \\ 0 \\ -v_L \end{bmatrix} = j \underbrace{\begin{bmatrix} 0 & K_{S,1} & K_{S,2} & \cdots & K_{S,N} & K_{S,L} \\ K_{S,1} & \omega L_1 + X_1 & K_{1,2} & \cdots & K_{1,N} & K_{1,L} \\ K_{S,2} & K_{1,2} & \omega L_2 + X_2 & \cdots & K_{2,N} & K_{2,L} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{S,N} & K_{1,N} & K_{2,N} & \cdots & \omega L_N + X_N & K_{N,L} \\ K_{S,L} & K_{1,L} & K_{2,L} & \cdots & K_{N,L} & 0 \end{bmatrix}}_{[\mathrm{Z}]} \begin{bmatrix} i_S \\ i_1 \\ i_2 \\ \vdots \\ i_N \\ i_L \end{bmatrix}. \qquad (2.26)$$

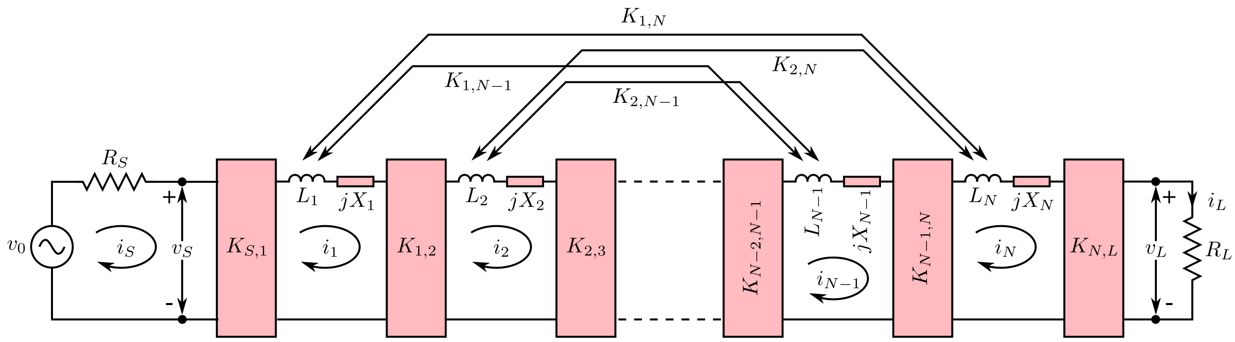


Figure 2.8: Series type LPP with source/load to inner resonator couplings.

Normalizing the above matrix gives

$$\begin{bmatrix} v_S \\ 0 \\ 0 \\ \vdots \\ 0 \\ -v_L \end{bmatrix} = j \underbrace{\begin{bmatrix} 0 & \frac{K_{S,1}}{\sqrt{L_1}} & \frac{K_{S,2}}{\sqrt{L_2}} & \cdots & \frac{K_{S,N}}{\sqrt{L_N}} & K_{S,L} \\ \frac{K_{S,1}}{\sqrt{L_1}} & \omega + \frac{X_1}{L_1} & \frac{K_{1,2}}{\sqrt{L_1 L_2}} & \cdots & \frac{K_{1,N}}{\sqrt{L_1 L_N}} & \frac{K_{1,L}}{\sqrt{L_1}} \\ \frac{K_{S,2}}{\sqrt{L_2}} & \frac{K_{1,2}}{\sqrt{L_1 L_2}} & \omega + \frac{X_2}{L_2} & \cdots & \frac{K_{2,N}}{\sqrt{L_2 L_N}} & \frac{K_{2,L}}{\sqrt{L_2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{S,N} & \frac{K_{1,N}}{\sqrt{L_1 L_N}} & \frac{K_{2,N}}{\sqrt{L_2 L_N}} & \cdots & \omega + \frac{X_N}{L_N} & \frac{K_{N,L}}{\sqrt{L_N}} \\ K_{S,L} & \frac{K_{1,L}}{\sqrt{L_1}} & \frac{K_{2,L}}{\sqrt{L_2}} & \cdots & \frac{K_{N,L}}{\sqrt{L_N}} & 0 \end{bmatrix}}_{[\mathrm{Z}^{\mathrm{norm}}]=j[[\mathrm{M}]+\omega[\mathrm{I}]]} \begin{bmatrix} i_S \\ i_1\sqrt{L_1} \\ i_2\sqrt{L_2} \\ \vdots \\ i_N\sqrt{L_N} \\ i_L \end{bmatrix}. \tag{2.27}$$

Re-writing (2.27) gives

$$\begin{aligned} \begin{bmatrix} i_S \\ i_1\sqrt{L_1} \\ i_2\sqrt{L_2} \\ \vdots \\ i_N\sqrt{L_N} \\ i_L \end{bmatrix} &= [\mathrm{Z}^{\mathrm{norm}}]^{-1} \begin{bmatrix} v_S \\ 0 \\ 0 \\ \vdots \\ 0 \\ -v_L \end{bmatrix} \\ \Rightarrow \begin{bmatrix} i_S \\ i_L \end{bmatrix} &= \begin{bmatrix} [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{1,1} & [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{1,N+2} \\ [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{N+2,1} & [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{N+2,N+2} \end{bmatrix} \begin{bmatrix} v_S \\ -v_L \end{bmatrix} \\ \Rightarrow \begin{bmatrix} i_S \\ -i_L \end{bmatrix} &= \begin{bmatrix} [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{1,1} & -[\mathrm{Z}^{\mathrm{norm}}]^{-1}_{1,N+2} \\ -[\mathrm{Z}^{\mathrm{norm}}]^{-1}_{N+2,1} & [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{N+2,N+2} \end{bmatrix} \begin{bmatrix} v_S \\ v_L \end{bmatrix} \end{aligned} \tag{2.28}$$

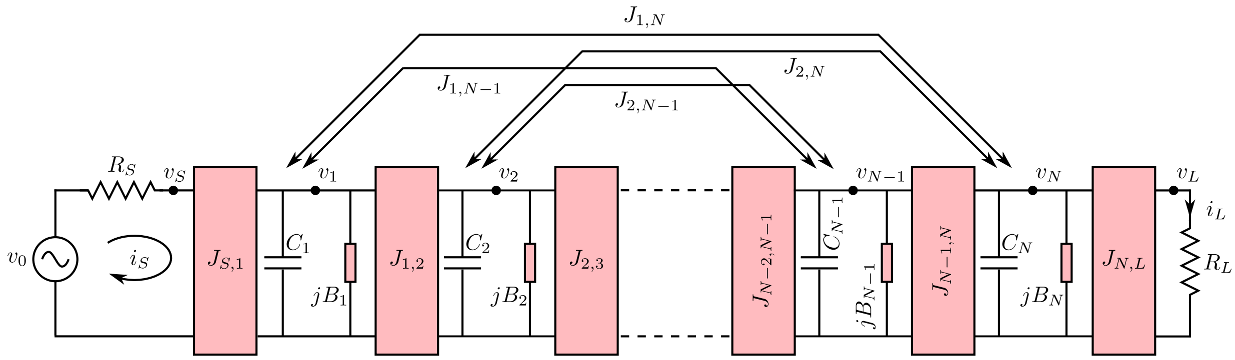


Figure 2.9: Shunt type LPP with source/load to inner resonator couplings.

2.6.1.2 Shunt Type LPP

From the duality principle and (2.28),

$$\begin{aligned} \begin{bmatrix} v_S \\ -v_L \end{bmatrix} &= \begin{bmatrix} [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{1,1} & -[\mathrm{Y}^{\mathrm{norm}}]^{-1}_{1,N+2} \\ -[\mathrm{Y}^{\mathrm{norm}}]^{-1}_{N+2,1} & [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{N+2,N+2} \end{bmatrix} \begin{bmatrix} i_S \\ i_L \end{bmatrix} \\ \Rightarrow \begin{bmatrix} v_S \\ v_L \end{bmatrix} &= \begin{bmatrix} [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{1,1} & [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{1,N+2} \\ [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{N+2,1} & [\mathrm{Y}^{\mathrm{norm}}]^{-1}_{N+2,N+2} \end{bmatrix} \begin{bmatrix} i_S \\ -i_L \end{bmatrix}, \end{aligned} \tag{2.29}$$

where

$$[\mathrm{Y}^{\mathrm{norm}}] = j \begin{bmatrix} 0 & \frac{J_{S,1}}{\sqrt{C_1}} & \frac{J_{S,2}}{\sqrt{C_2}} & \cdots & \frac{J_{S,N}}{\sqrt{C_N}} & J_{S,L} \\ \frac{J_{S,1}}{\sqrt{C_1}} & \omega + \frac{B_1}{C_1} & \frac{J_{1,2}}{\sqrt{C_1 C_2}} & \cdots & \frac{J_{1,N}}{\sqrt{C_1 C_N}} & \frac{J_{1,L}}{\sqrt{C_1}} \\ \frac{J_{S,2}}{\sqrt{C_2}} & \frac{J_{1,2}}{\sqrt{C_1 C_2}} & \omega + \frac{B_2}{C_2} & \cdots & \frac{J_{2,N}}{\sqrt{C_2 C_N}} & \frac{J_{2,L}}{\sqrt{C_2}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ J_{S,N} & \frac{J_{1,N}}{\sqrt{C_1 C_N}} & \frac{J_{2,N}}{\sqrt{C_2 C_N}} & \cdots & \omega + \frac{B_N}{C_N} & \frac{J_{N,L}}{\sqrt{C_N}} \\ J_{S,L} & \frac{J_{1,L}}{\sqrt{C_1}} & \frac{J_{2,L}}{\sqrt{C_2}} & \cdots & \frac{J_{N,L}}{\sqrt{C_N}} & 0 \end{bmatrix}. \tag{2.30}$$

2.6.2 Synthesis of the General $N + 2$ Coupling Matrix

2.6.2.1 Series Type LPP

From (2.28),

$$\begin{aligned} [\mathrm{Z}^{\mathrm{norm}}]^{-1}_{11} &= y_{11} \\ \Rightarrow [[\mathrm{M}] + \omega\,[\mathrm{I}]]^{-1}_{11} &= j y_{11}. \end{aligned} \tag{2.31}$$

Following the theory presented in Sec. 2.5.2.1,

$$\begin{aligned} \sum_{k=1}^{N+2} \frac{T_{1k}^2}{\omega+\lambda_k} &= jy_{11} \\ \Rightarrow \sum_{k=1}^{N+2} \frac{T_{1k}^2}{\omega+\lambda_k} &= \frac{j}{R_S}\frac{(\mathbb{EF}_- + \mathbb{EF}_{-*})}{(\mathbb{EF}_+ - \mathbb{EF}_{+*})}, \end{aligned} \tag{2.32}$$

where, $[\mathrm{M}] = [\mathrm{T}]\,[\Lambda]\,[\mathrm{T}]^t$. Similarly,

$$\begin{aligned} -\left[\mathrm{Z}^{\mathrm{norm}}\right]^{-1}_{1,N+2} &= y_{21} \\ \Rightarrow \sum_{k=1}^{N+2} \frac{T_{1k}T_{N+2,k}}{\omega+\lambda_k} &= \frac{j}{\sqrt{R_S R_L}}\frac{2P}{\varepsilon\,(\mathbb{EF}_+ - \mathbb{EF}_{+*})}. \end{aligned} \tag{2.33}$$

2.6.2.2 Shunt Type LPP

From (2.29) and (1.18),

$$\begin{aligned} \left[\mathrm{Y}^{\mathrm{norm}}\right]^{-1}_{11} &= z_{11} \\ \Rightarrow \sum_{k=1}^{N+2} \frac{T_{1k}^2}{\omega+\lambda_k} &= jz_{11} \\ \Rightarrow \sum_{k=1}^{N+2} \frac{T_{1k}^2}{\omega+\lambda_k} &= jR_S\frac{(\mathbb{EF}_+ + \mathbb{EF}_{+*})}{(\mathbb{EF}_- - \mathbb{EF}_{-*})}. \end{aligned} \tag{2.34}$$

where, $[\mathrm{M}] = [\mathrm{T}]\,[\Lambda]\,[\mathrm{T}]^t$. Similarly,

$$\begin{aligned} \left[\mathrm{Y}^{\mathrm{norm}}\right]^{-1}_{1,N+2} &= z_{21} \\ \Rightarrow \sum_{k=1}^{N+2} \frac{T_{1k}T_{N+2,k}}{\omega+\lambda_k} &= j\sqrt{R_S R_L}\frac{2P}{\varepsilon\,(\mathbb{EF}_- - \mathbb{EF}_{-*})}. \end{aligned} \tag{2.35}$$

2.6.3 Synthesis of the $N+2$ Transversal Matrix

Till now, all the synthesis techniques needed the Gram-Schmidt orthonormalization step. This step can be avoided if one starts with a simpler transversal LPP configuration shown in Fig. 2.10. In this configuration, no coupling exists between the resonators. Only coupling that exists for each resonator is the corresponding interaction with source/load. ABCD parameter matrix corresponding to the highlighted two port section is given as

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix}_N &= \begin{bmatrix} 0 & \frac{j}{J_{S,N}} \\ jJ_{S,N} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ j\omega C_N + jB_N & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{j}{J_{N,L}} \\ jJ_{N,L} & 0 \end{bmatrix} \\ &= -\begin{bmatrix} \frac{J_{N,L}}{J_{S,N}} & \left(\frac{j\omega C_N + jB_N}{J_{S,N}J_{N,L}}\right) \\ 0 & \frac{J_{S,N}}{J_{N,L}} \end{bmatrix}. \end{aligned}$$

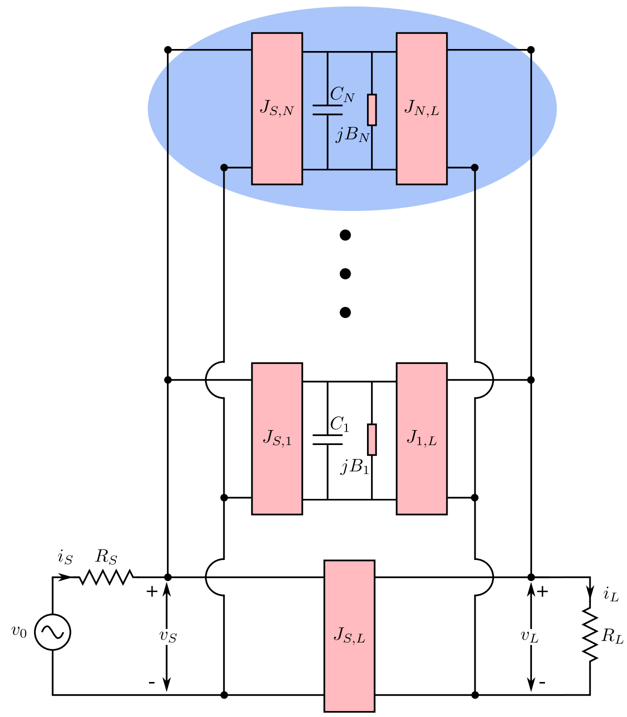


Figure 2.10: Canonical transversal LPP configuration.

Converting ABCD parameters to Y parameters gives

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}_N = \frac{1}{b}\begin{bmatrix} d & -1 \\ -1 & a \end{bmatrix}$$
$$= \frac{1}{j\omega C_N + jB_N}\begin{bmatrix} J_{S,N}^2 & J_{S,N}J_{N,L} \\ J_{S,N}J_{N,L} & J_{N,L}^2 \end{bmatrix}.$$

Since all two port sections are connected parallelly, overall Y parameter matrix is given as

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}_{\text{total}} = \begin{bmatrix} 0 & \frac{j}{J_{S,L}} \\ jJ_{S,L} & 0 \end{bmatrix} + \sum_{k=1}^{N} \frac{1}{j\omega C_k + jB_k}\begin{bmatrix} J_{S,k}^2 & J_{S,k}J_{k,L} \\ J_{S,k}J_{k,L} & J_{k,L}^2 \end{bmatrix}. \tag{2.36}$$

From (2.36) and (1.17)

$$\frac{(\mathbb{EF}_- + \mathbb{EF}_{-*})}{R_S\,(\mathbb{EF}_+ - \mathbb{EF}_{+*})} = \sum_{k=1}^{N} \frac{J_{S,k}^2}{j\omega C_k + jB_k} \text{ and}$$
$$-\frac{1}{\sqrt{R_S R_L}}\frac{2P}{\varepsilon\,(\mathbb{EF}_+ - \mathbb{EF}_{+*})} = \frac{j}{J_{S,L}} + \sum_{k=1}^{N} \frac{J_{S,k}J_{k,L}}{j\omega C_k + jB_k}. \tag{2.37}$$

So, one can obtain first and last rows (and columns), and all the diagonal elements of (2.30) by equating poles and residues on both sides of (2.37). In addition, all other elements are zero for transveresal prototype.