Waveguides

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Outline

1. Helmholtz Wave Equation
2. TE, TM, and TEM Modes – Rect
3. Rectangular Waveguide
4. TE, TM, and TEM Modes – Cyl
5. Cylindrical Waveguide
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1. Helmholtz Wave Equation
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**Helmholtz Wave Equation**

In a source-less dielectric medium,

\[ \nabla \cdot \vec{D} = 0 \]
\[ \nabla \cdot \vec{B} = 0 \]
\[ \nabla \times \vec{H} = j\omega \vec{D} = j\omega \varepsilon \vec{E} \]  \( (1) \)
\[ \nabla \times \vec{E} = -j\omega \vec{B} = -j\omega \mu \vec{H} \]  \( (2) \)

Taking curl of (2) gives

\[ \nabla \times (\nabla \times \vec{E}) = \nabla \times (-j\omega \mu \vec{H}) \]
\[ \Rightarrow \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -j\omega \mu (\nabla \times \vec{H}) \]
\[ \Rightarrow \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -j\omega \mu (j\omega \varepsilon \vec{E}) \]
\[ \Rightarrow \nabla^2 \vec{E} = \nabla (\nabla \cdot \vec{E}) - \omega^2 \mu \varepsilon \vec{E} \]
\[ \Rightarrow \nabla^2 \vec{E} = \vec{0} - \omega^2 \mu \varepsilon \vec{E} \]  \( (3) \)

Similarly, it can be proved that

\[ \nabla^2 \vec{H} = -\omega^2 \mu \varepsilon \vec{H}. \]  \( (4) \)
Solution of Helmholtz Equation (in Cartesian System)

Vector Helmholtz equation can be decomposed as shown below:

\[
\nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0
\]

Since all the differential equations are similar, let’s solve just one equation using variable-separable method. If \( E_{xs} \) can be decomposed into

\[
E_x = A(x) B(y) C(z)
\]

then substituting the above equation into Helmholtz equation gives

\[
\nabla^2 E_x + \omega^2 \mu \varepsilon E_x = 0
\]

\[
\Rightarrow \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + \omega^2 \mu \varepsilon E_x = 0
\]

\[
\Rightarrow B(y) C(z) \frac{\partial^2 A}{\partial x^2} + A(x) C(z) \frac{\partial^2 B}{\partial y^2} + A(x) B(y) \frac{\partial^2 C}{\partial z^2} + \omega^2 \mu \varepsilon A(x) B(y) C(z) = 0
\]

\[
\Rightarrow \frac{1}{A(x)} \frac{\partial^2 A}{\partial x^2} + \frac{1}{B(y)} \frac{\partial^2 B}{\partial y^2} + \frac{1}{C(z)} \frac{\partial^2 C}{\partial z^2} - \gamma^2 = 0
\]
Solution of Helmholtz Equation ... Contd

\[ \frac{\partial^2 A}{\partial x^2} + \frac{1}{B(y)} \frac{\partial^2 B}{\partial y^2} + \frac{1}{C(z)} \frac{\partial^2 C}{\partial z^2} - \gamma^2 = 0 \]

\[ \Rightarrow \frac{1}{A(x)} \frac{\partial^2 A}{\partial x^2} + \frac{1}{B(y)} \frac{\partial^2 B}{\partial y^2} + \frac{1}{C(z)} \frac{\partial^2 C}{\partial z^2} - \gamma^2 = 0 \]  

(5)

The above equation can be decomposed into 3 separate equations:

\[ \frac{1}{A(x)} \frac{\partial^2 A}{\partial x^2} - \gamma_x^2 = 0 \]

\[ \frac{1}{B(y)} \frac{\partial^2 B}{\partial y^2} - \gamma_y^2 = 0 \]

\[ \frac{1}{C(z)} \frac{\partial^2 C}{\partial z^2} - \gamma_z^2 = 0 \]

It is sufficient to solve only one of the above equations and it’s solution is given as

\[ \frac{\partial^2 A}{\partial x^2} - \gamma_x^2 A(x) = 0 \]

\[ \Rightarrow A(x) = L_1 e^{\gamma_x x} + L_2 e^{-\gamma_x x} = L^- e^{\gamma_x x} + L^+ e^{-\gamma_x x} \]  

(6)
Solution of Helmholtz Equation ... Contd

So, finally $E_x$ is given as

$$E_x = (L^- e^{\gamma xx} + L^+ e^{-\gamma xx}) (M^- e^{\gamma yy} + M^+ e^{-\gamma yy}) (N^- e^{\gamma zz} + N^+ e^{-\gamma zz})$$

(7)

with the condition that

$$\gamma_x^2 + \gamma_y^2 + \gamma_z^2 = \gamma^2.$$  

(8)
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Maxwell’s Equations in Free Space

\[ \nabla \times \vec{E} = -j\omega \mu \vec{H} \]

\[ \frac{\partial E_z}{\partial y} + j\beta_z E_y = -j\omega \mu H_x \]

\[ -j\beta_z E_x - \frac{\partial E_z}{\partial x} = -j\omega \mu H_y \]

\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega \mu H_z \]

\[ \nabla \times \vec{H} = j\omega \varepsilon \vec{E} \]

\[ \frac{\partial H_z}{\partial y} + j\beta_z H_y = j\omega \varepsilon E_x \]

\[ -j\beta_z H_x - \frac{\partial H_z}{\partial x} = j\omega \varepsilon E_y \]

\[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega \mu \varepsilon E_z \]

\[ H_x = \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega\varepsilon \frac{\partial E_z}{\partial y} - \beta_z \frac{\partial H_z}{\partial x} \right) \]

\[ H_y = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega\varepsilon \frac{\partial E_z}{\partial x} + \beta_z \frac{\partial H_z}{\partial y} \right) \]

\[ E_x = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial x} + \omega\mu \frac{\partial H_z}{\partial y} \right) \]

\[ E_y = \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial E_z}{\partial y} + \omega\mu \frac{\partial H_z}{\partial x} \right) \]

The above equations indicate the fact that if longitudinal components \( E_z \) & \( H_z \) are known, one can get all the transversal components.
Helmholtz Wave Equation

TE, TM, and TEM Modes – Rectangular Waveguide

TE, TM, and TEM Modes – Cylindrical Waveguide

TE Modes – $E_z = 0$

\[
\begin{align*}
H_x &= \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial y} - \beta_z \frac{\partial H_z}{\partial x} \right) = \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial H_z}{\partial x} \right) \\
H_y &= \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial x} + \beta_z \frac{\partial H_z}{\partial y} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial H_z}{\partial y} \right) \\
E_x &= \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \mu \frac{\partial H_z}{\partial y} \right) \\
E_y &= \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right) = \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega \mu \frac{\partial H_z}{\partial x} \right)
\end{align*}
\]

Characteristic impedance of TE wave is given as

\[
Z_{0\text{TE}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega \mu}{\beta_z}
\]
TM Modes – \( H_z = 0 \)

\[
\begin{align*}
H_x &= \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial y} - \beta_z \frac{\partial H_z}{\partial x} \right) = \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial y} \right) \\
H_y &= \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial x} + \beta_z \frac{\partial H_z}{\partial y} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial x} \right) \\
E_x &= \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial x} \right) \\
E_y &= \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right) = \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial E_z}{\partial y} \right)
\end{align*}
\]

Characteristic impedance of TM waved is given as

\[
Z_{0\text{TM}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta_z}{\omega \varepsilon}
\]
TEM Modes – \( E_z = 0 \) \& \( H_z = 0 \)

\[
\begin{align*}
H_x &= \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega \epsilon \frac{\partial E_z}{\partial y} - \beta_z \frac{\partial H_z}{\partial x} \right) \\
H_y &= \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \epsilon \frac{\partial E_z}{\partial x} + \beta_z \frac{\partial H_z}{\partial y} \right) \\
E_x &= \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right) \\
E_y &= \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right)
\end{align*}
\]

If longitudinal components \( E_z \& H_z \) are zeros, all the transversal components become zero ... then the wave does not exist at all !!!

So, something must be wrong ... isn’t it !!
TEM Modes – \( E_z = 0 \) & \( H_z = 0 \)

\[
H_x = \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial y} - \beta_z \frac{\partial H_z}{\partial x} \right)
\]

\[
H_y = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial x} + \beta_z \frac{\partial H_z}{\partial y} \right)
\]

\[
E_x = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right)
\]

\[
E_y = \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right)
\]

Transverse components can exist only when \( \beta_z = \beta_0 \). In other words, \( \beta_x = 0 \) and \( \beta_y = 0 \).
TEM Modes – $Z_{0}^{TEM}$

Characteristic impedance of TE and TM waves are given as

$$Z_{0}^{\text{TE}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\omega \mu}{\beta_z}$$

$$Z_{0}^{\text{TM}} = \frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{\beta_z}{\omega \varepsilon}$$

Then what is the Characteristic impedance of TEM wave?

\[ \downarrow \]

Since TEM mode is TE as well as TM, we can actually use any one of the above equations ... also, for TEM waves $\beta_z = \beta_0$,

$$Z_{0}^{\text{TE}} = \frac{\omega \mu}{\beta_z} = \frac{\omega \mu}{\beta_0} = \frac{\omega \mu}{\omega \sqrt{\mu \varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_{0}^{\text{TM}} = \frac{\beta_z}{\omega \varepsilon} = \frac{\beta_0}{\omega \varepsilon} = \frac{\omega \sqrt{\mu \varepsilon}}{\omega \varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}$$

Have you noticed that the Characteristic impedance of TEM wave is independent of waveguide dimensions ?!
Existence of TEM Modes

For TEM waves $E$ and $H$ fields exist in transverse plane only (since $E_z = 0$ & $H_z = 0$). Also, from the maxwell equation $\nabla \cdot \vec{B} = 0$, we know that magnetic fields are rotational fields. So, line integral of $\vec{H}$ around any arbitrary closed loop $\oint_C \vec{H} \cdot d\vec{l}$ has to be non-zero. However, from Stokes’ theorem and Ampere’s law, we know that

$$\oint_C \vec{H} \cdot d\vec{l} = \iint (\nabla \times \vec{H}) \cdot d\vec{s} = \iint \vec{J}_e \cdot d\vec{s}.$$

Since $\oint_C H \cdot d\vec{l} \neq 0$, $\vec{J}_e$ has to be a non-zero value. However, we know that $\vec{J}_e$ is conduction current ...
So, we need at least one conductor inside the waveguide for TEM modes to exist.
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From (7), similar to $E_x$, $E_z$ also can be written as

$$E_z = (L^-e^{\gamma_xx} + L^+e^{-\gamma_xx}) (M^-e^{\gamma_yy} + M^+e^{-\gamma_yy}) (N^-e^{\gamma_zz} + N^+e^{-\gamma_zz}).$$  \hfill (9)

If the wave is propagating along $+z$ direction, $N^- = 0$. Remaining constants can be derived using boundary conditions. Since $E_z$ is tangential to all the four surfaces, boundary conditions are

$$\begin{cases} 
E_z = 0, & \text{when } x = 0 \& x = a \\
E_z = 0, & \text{when } y = 0 \& y = b 
\end{cases}. \hfill (10)$$
Longitudinal Component – $E_z$

First let us see the boundary conditions imposed by $x = 0$ and $x = a$. Since $E_z = 0$, when $x = 0$,

$$L^- + L^+ = 0 \implies L^- = -L^+. \quad (11)$$

Similarly $E_z = 0$, when $x = a$, so using the above equation one gets

$$L^- e^{\gamma_x a} + L^+ e^{-\gamma_x a} = 0 \implies L^+ (e^{-\gamma_x a} - e^{\gamma_x a}) = 0 \implies \sinh \gamma_x a = 0.$$

Hence, from the boundary conditions, we have $\gamma_x a = \sinh^{-1} (0) = j \sin^{-1} (0) = jm\pi$, where $m = 0, \pm 1, \pm 2, \ldots$. So,

$$\gamma_x = j \frac{m\pi}{a} = 0 + j\beta_x. \quad (12)$$

Similarly, from boundary conditions at $y = 0$ and $y = b$, we can derive that

$$M^- = -M^+ \quad \text{and} \quad \gamma_y = j \frac{n\pi}{b} = 0 + j\beta_y, \text{ where } n = 0, \pm 1, \pm 2, \ldots \quad (13)$$
Longitudinal Component – $E_z$

Substituting (11), (12), and (13) into (9) gives

$$E_z = \left( L^− e^{\gamma_xx} + L^+ e^{−\gamma_xx} \right) \left( M^− e^{\gamma_yy} + M^+ e^{−\gamma_yy} \right) \left( N^− e^{\gamma_zz} + N^+ e^{−\gamma_zz} \right)$$

$$= \left( -L^+ e^{\gamma_xx} + L^+ e^{−\gamma_xx} \right) \left( -M^+ e^{\gamma_yy} + M^+ e^{−\gamma_yy} \right) \left( 0 + N^+ e^{−\gamma_zz} \right)$$

$$= \left( -L^+ \right) \left( -M^+ \right) \left( N^+ \right) \left( e^{\gamma_xx} - e^{−\gamma_xx} \right) \left( e^{\gamma_yy} - e^{−\gamma_yy} \right) \left( e^{−\gamma_zz} \right)$$

$$= \left( 2L^+ \right) \left( 2M^+ \right) \left( N^+ \right) \left( j^2 \right) \sinh (\gamma_xx) \sinh (\gamma_yy) e^{−\gamma_zz}$$

$$= \left( 2L^+ \right) \left( 2M^+ \right) \left( N^+ \right) \left( j^2 \right) \sin (\beta_xx) \sin (\beta_yy) e^{−\gamma_zz}, \quad \therefore \sinh (jx) = j \sin x$$

$$= A \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) e^{−\gamma_zz}.$$  \hspace{1cm} \text{(14)}

We also know that $\beta_x^2 + \beta_y^2 + \beta_z^2 = \beta_0^2$. So, $\gamma_z = j\beta_z$ where

$$\beta_z = \sqrt{\beta_0^2 - \beta_x^2 - \beta_y^2} = \sqrt{\beta_0^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2}.$$  \hspace{1cm} \text{(15)}
When electric field is zero, magnetic field is maximum. So, for magnetic fields, sin function simply becomes cos function. So, we do not have to derive expression for $H_z$ once again. So, $E_z$ and $H_z$ are simply given as

\begin{align}
E_z &= A \sin \left( \frac{m\pi}{a} x \right) \sin \left( \frac{n\pi}{b} y \right) e^{-\gamma z} \quad (16) \\
H_z &= B \cos \left( \frac{m\pi}{a} x \right) \cos \left( \frac{n\pi}{b} y \right) e^{-\gamma z} \quad (17)
\end{align}
TE Modes in Rectangular Waveguide – $E_z = 0$

Now that we have derived expressions for longitudinal components, we can get transversal components by using the formulas for TE waves:

$$H_x = \frac{j}{\beta_0^2 - \beta_z^2} \left(-\beta_z \frac{\partial H_z}{\partial x}\right) = B \frac{-j\beta_z}{\beta_0^2 - \beta_z^2} \left[-\frac{m\pi}{a} \times \sin \left(\frac{m\pi}{a} x\right) \cos \left(\frac{n\pi}{b} y\right) e^{-\gamma z}\right]$$

$$H_y = \frac{-j}{\beta_0^2 - \beta_z^2} \left(\beta_z \frac{\partial H_z}{\partial y}\right) = B \frac{-j\beta_z}{\beta_0^2 - \beta_z^2} \left[-\frac{n\pi}{b} \times \cos \left(\frac{m\pi}{a} x\right) \sin \left(\frac{n\pi}{b} y\right) e^{-\gamma z}\right]$$

$$E_x = \frac{-j}{\beta_0^2 - \beta_z^2} \left(\omega \mu \frac{\partial H_z}{\partial y}\right) = B \frac{-j\omega \mu}{\beta_0^2 - \beta_z^2} \left[-\frac{n\pi}{b} \times \cos \left(\frac{m\pi}{a} x\right) \sin \left(\frac{n\pi}{b} y\right) e^{-\gamma z}\right]$$

$$E_y = \frac{j}{\beta_0^2 - \beta_z^2} \left(\omega \mu \frac{\partial H_z}{\partial x}\right) = B \frac{j\omega \mu}{\beta_0^2 - \beta_z^2} \left[-\frac{m\pi}{a} \times \sin \left(\frac{m\pi}{a} x\right) \cos \left(\frac{n\pi}{b} y\right) e^{-\gamma z}\right]$$

since $H_z = B \cos \left(\frac{m\pi}{a} x\right) \cos \left(\frac{n\pi}{b} y\right) e^{-\gamma z}$. 
TM Modes in Rectangular Waveguide

Similar to TE mode, TM mode’s transversal components are given as shown below:

\[
H_x = \frac{j}{\beta_0^2 - \beta_z^2} \left( \omega e \frac{\partial E_z}{\partial y} \right) = A \frac{j \omega e}{\beta_0^2 - \beta_z^2} \left[ \frac{n \pi}{b} \times \sin \left( \frac{m \pi}{a} x \right) \cos \left( \frac{n \pi}{b} y \right) e^{-\gamma_z z} \right]
\]

\[
H_y = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega e \frac{\partial E_z}{\partial x} \right) = A \frac{-j \omega e}{\beta_0^2 - \beta_z^2} \left[ \frac{m \pi}{a} \times \cos \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-\gamma_z z} \right]
\]

\[
E_x = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial x} \right) = A \frac{-j \beta_z}{\beta_0^2 - \beta_z^2} \left[ \frac{m \pi}{a} \times \cos \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-\gamma_z z} \right]
\]

\[
E_y = \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial E_z}{\partial y} \right) = A \frac{-j \beta_z}{\beta_0^2 - \beta_z^2} \left[ \frac{n \pi}{b} \times \sin \left( \frac{m \pi}{a} x \right) \cos \left( \frac{n \pi}{b} y \right) e^{-\gamma_z z} \right]
\]

since \( E_z = A \sin \left( \frac{m \pi}{a} x \right) \sin \left( \frac{n \pi}{b} y \right) e^{-\gamma_z z} \).
If you are fed up with the all the equations given in this PDF file, you may feel free to refer to much simpler explanation given in another PDF file written on rectangular waveguides. There, you can also find formulas for $\lambda_g$, $\omega_{cutoff}$, etc.

Modes in Rectangular Waveguide – Intuitive Explanation
Parallel Plate Waveguide

All the equations derived for rectangular waveguides can also be used for parallel plate waveguide. After all, parallel plate waveguide is nothing but a rectangular waveguide with $a \to \infty$.

One major difference though is, in parallel plate waveguide in addition to TE and TM modes, TEM modes also exist ... do you know why?
Once again, the equations derived for rectangular waveguides can be used for free space too. It is because, free space waveguide is nothing but a rectangular waveguide with $a \to \infty$ and $b \to \infty$.

One major difference though is, similar to parallel plate waveguide, in free space, TEM modes also exist ... do you know why?
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Maxwell’s Equations in Free Space

\[ \nabla \times \vec{E} = -j\omega \mu \vec{H} \]

\[ \frac{1}{\rho} \left( \frac{\partial E_z}{\partial \phi} - \rho \frac{\partial E_\phi}{\partial z} \right) = -j\omega \mu H_\rho \]
\[ \left( \frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} \right) = -j\omega \mu H_\phi \]
\[ \frac{1}{\rho} \left( \frac{\partial (\rho E_\phi)}{\partial \rho} - \frac{\partial E_\rho}{\partial \phi} \right) = -j\omega \mu H_z \]

\[ \nabla \times \vec{H} = j\omega \epsilon \vec{E} \]

\[ \frac{1}{\rho} \left( \frac{\partial H_\rho}{\partial \phi} - \rho \frac{\partial H_\phi}{\partial z} \right) = j\omega \epsilon E_\rho \]
\[ \left( \frac{\partial H_\rho}{\partial z} - \frac{\partial H_\rho}{\partial \rho} \right) = j\omega \epsilon E_\phi \]
\[ \frac{1}{\rho} \left( \frac{\partial (\rho H_\phi)}{\partial \rho} - \frac{\partial H_\rho}{\partial \phi} \right) = j\omega \epsilon E_z \]

\[ E_\rho = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{\omega \mu}{\rho} \frac{\partial H_z}{\partial \phi} \right) \]
\[ E_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \frac{\partial E_z}{\partial \phi} - \frac{\omega \mu}{\rho} \frac{\partial H_z}{\partial \rho} \right) \]
\[ H_\rho = \frac{j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \epsilon}{\rho} \frac{\partial E_z}{\partial \rho} - \frac{\beta_z}{\rho} \frac{\partial H_z}{\partial \phi} \right) \]
\[ H_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \epsilon}{\rho} \frac{\partial E_z}{\partial \rho} + \frac{\beta_z}{\rho} \frac{\partial H_z}{\partial \phi} \right) \]

Once again, the above equations indicate the fact that if longitudinal components $E_z$ & $H_z$ are known, one can get all the transversal components.
TE Modes – $E_z = 0$

\[ E_\rho = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial \rho} + \frac{\omega \mu}{\rho} \frac{\partial H_z}{\partial \phi} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \mu \frac{\partial H_z}{\partial \phi} \right) \]

\[ E_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \frac{\partial E_z}{\partial \phi} - \omega \mu \frac{\partial H_z}{\partial \rho} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( -\omega \mu \frac{\partial H_z}{\partial \rho} \right) \]

\[ H_\rho = \frac{j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \varepsilon}{\rho} \frac{\partial E_z}{\partial \phi} - \beta_z \frac{\partial H_z}{\partial \rho} \right) = \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial H_z}{\partial \rho} \right) \]

\[ H_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial \rho} + \frac{\beta_z}{\rho} \frac{\partial H_z}{\partial \phi} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial H_z}{\partial \phi} \right) \]

Characteristic impedance of TE wave is given as

\[ Z_{0,TE}^{TE} = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho} = \frac{\omega \mu}{\beta_z} \]
Helmholtz Wave Equation

TM Modes – $H_z = 0$

\[
E_\rho = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial \rho} + \frac{\omega \mu}{\rho} \frac{\partial H_z}{\partial \phi} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial \rho} \right)
\]

\[
E_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \frac{\partial E_z}{\partial \phi} - \omega \mu \frac{\partial H_z}{\partial \rho} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \frac{\partial E_z}{\partial \phi} \right)
\]

\[
H_\rho = \frac{j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \epsilon}{\rho} \frac{\partial E_z}{\partial \phi} - \beta_z \frac{\partial H_z}{\partial \rho} \right) = \frac{j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \epsilon}{\rho} \frac{\partial E_z}{\partial \phi} \right)
\]

\[
H_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \epsilon}{\rho} \frac{\partial E_z}{\partial \rho} + \beta_z \frac{\partial H_z}{\partial \phi} \right) = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \epsilon}{\rho} \frac{\partial E_z}{\partial \rho} \right)
\]

Characteristic impedance of TE waved is given as

\[
Z_{0}^{TM} = \frac{E_\rho}{H_\phi} = -\frac{E_\phi}{H_\rho} = \frac{\beta_z}{\omega \epsilon}
\]
TEM Modes – $Z_{0}^{\text{TEM}}$

Characteristic impedance of TE and TM waves are given as

$$Z_{0}^{\text{TE}} = \frac{E_{\rho}}{H_{\phi}} = -\frac{E_{\phi}}{H_{\rho}} = \frac{\omega \mu}{\beta_{z}}$$

$$Z_{0}^{\text{TM}} = \frac{E_{\rho}}{H_{\phi}} = -\frac{E_{\phi}}{H_{\rho}} = \frac{\beta_{z}}{\omega \varepsilon}$$

Then what is the Characteristic impedance of TEM wave?

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Since TEM mode is TE as well as TM, we can actually use any one of the above equations ... also, for TEM waves $\beta_{z} = \beta_{0}$,

$$Z_{0}^{\text{TE}} = \frac{\omega \mu}{\beta_{z}} = \frac{\omega \mu}{\beta_{0}} = \frac{\omega \mu}{\omega \sqrt{\mu \varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_{0}^{\text{TM}} = \frac{\beta_{z}}{\omega \varepsilon} = \frac{\beta_{0}}{\omega \varepsilon} = \frac{\omega \sqrt{\mu \varepsilon}}{\omega \varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}$$

Have you noticed that the Characteristic impedance of TEM wave is independent of waveguide dimensions ?!
Outline

1. Helmholtz Wave Equation
2. TE, TM, and TEM Modes – Rect
3. Rectangular Waveguide
4. TE, TM, and TEM Modes – Cyl
5. Cylindrical Waveguide
Helmholtz Wave Equation - Cylindrical Coordinate System

Since all the electromagnetic field components have to satisfy the Helmholtz wave equation, one can write the wave equation for \( E_z \) as

\[
\nabla^2 E_z + \omega^2 \mu \varepsilon E_z = 0
\]

\[
\Rightarrow \left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial E_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} \right] + \omega^2 \mu \varepsilon E_z = 0.
\]

Solving the above differential equation gives

\[
E_z = [A J_n (\beta_c \rho) + B Y_n (\beta_c \rho)] \left[ C \sin (n \phi) + D \cos (n \phi) \right] \left[ E e^{-j\beta_z z} + F e^{j\beta_z z} \right]
\]

(18)
In (18), the term $Y_n (\beta_c \rho) \rightarrow \infty$ at $\rho = 0$; So, this term is physically unacceptable for cylindrical waveguides (however, this term is required for coaxial lines) ... also, if we assume that the wave is propagating in $+z$ direction, then $F = 0$ ... So,

$$E_z = AJ_n (\beta_c \rho) [C \sin (n\phi) + D \cos (n\phi)] e^{-j\beta_z z}. \quad (19)$$

Similarly, one can write equation for $H_z$ as

$$H_z = BJ_n (\beta_c \rho) [C' \sin (n\phi) + D' \cos (n\phi)] e^{-j\beta_z z}. \quad (20)$$
Cylindrical Waveguide – TE modes

\[ E_\rho = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \mu}{\rho} \frac{\partial H_z}{\partial \phi} \right) = B \frac{-j}{\beta_0^2 - \beta_z^2} \left[ \frac{\omega \mu}{\rho} \left[ J_n (\beta_c \rho) \left[ nC' \cos (n \phi) - nD' \sin (n \phi) \right] e^{-j \beta_z z} \right] \right] \]

\[ E_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( -\omega \mu \frac{\partial H_z}{\partial \rho} \right) = B \frac{-j}{\beta_0^2 - \beta_z^2} \left[ -\omega \mu \left[ \beta_c J_n' (\beta_c \rho) \left[ C' \sin (n \phi) + D' \cos (n \phi) \right] e^{-j \beta_z z} \right] \right] \]

\[ H_\rho = \frac{j}{\beta_0^2 - \beta_z^2} \left( -\beta_z \frac{\partial H_z}{\partial \rho} \right) = B \frac{j}{\beta_0^2 - \beta_z^2} \left[ -\beta_z \left[ \beta_c J_n' (\beta_c \rho) \left[ C' \sin (n \phi) + D' \cos (n \phi) \right] e^{-j \beta_z z} \right] \right] \]

\[ H_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \frac{\partial H_z}{\partial \phi} \right) = B \frac{-j}{\beta_0^2 - \beta_z^2} \left[ \frac{\beta_z}{\rho} \left[ J_n (\beta_c \rho) \left[ nC' \cos (n \phi) - nD' \sin (n \phi) \right] e^{-j \beta_z z} \right] \right] \]

where \( H_z = BJ_n (\beta_c \rho) \left[ C' \sin (n \phi) + D' \cos (n \phi) \right] e^{-j \beta_z z} \).

Cutoff propagation constant can be derived from the above equations by using boundary conditions. Since \( E_\phi \) is tangential to the boundary of the cylindrical waveguide, boundary conditions are

\[ \begin{cases} 
E_\phi = 0, & \text{when } \rho = a. 
\end{cases} \]  

(21)

So, boundary condition for TE modes is

\[ J'_n (\beta_c a) = 0. \]  

(22)
Cylindrical Waveguide – TM modes

\[
E_\rho = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \frac{\partial E_z}{\partial \rho} \right) = A \frac{-j}{\beta_0^2 - \beta_z^2} \left( \beta_z \left[ \beta_c J_n' (\beta_c \rho) [C \sin (n\phi) + D \cos (n\phi)] e^{-j\beta_z z} \right] \right)
\]

\[
E_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \frac{\partial E_z}{\partial \phi} \right) = A \frac{-j}{\beta_0^2 - \beta_z^2} \left( \frac{\beta_z}{\rho} \left[ J_n (\beta_c \rho) [nC \cos (n\phi) - nD' \sin (n\phi)] e^{-j\beta_z z} \right] \right)
\]

\[
H_\rho = \frac{j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \varepsilon}{\rho} \frac{\partial E_z}{\partial \phi} \right) = A \frac{j}{\beta_0^2 - \beta_z^2} \left( \frac{\omega \varepsilon}{\rho} \left[ J_n (\beta_c \rho) [nC \cos (n\phi) - nD' \sin (n\phi)] e^{-j\beta_z z} \right] \right)
\]

\[
H_\phi = \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \frac{\partial E_z}{\partial \rho} \right) = A \frac{-j}{\beta_0^2 - \beta_z^2} \left( \omega \varepsilon \left[ \beta_c J_n' (\beta_c \rho) [C \sin (n\phi) + D \cos (n\phi)] e^{-j\beta_z z} \right] \right)
\]

where \( E_z = AJ_n (\beta_c \rho) [C \sin (n\phi) + D \cos (n\phi)] e^{-j\beta_z z} \).

Cutoff propagation constant can be derived from the above equations by using boundary conditions. Since \( E_\phi \) is tangential to the boundary of the cylindrical waveguide, boundary conditions are

\[
\begin{cases} 
E_\phi = 0, & \text{when } \rho = a.
\end{cases}
\]

(23)

So, boundary condition for TM modes is

\[
J_n (\beta_c a) = 0.
\]

(24)